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Ergodic and Geometric Theory of Conservative and Hamiltonian Flows

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We learn to do things by doing the things we are learning to do.

Aristotle

Aos meus pais
e ao meu marido.

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RESUMO

Esta tese contém resultados que contribuem para o desenvolvimento da teoria da *dinâmica conservativa* e da *dinâmica Hamiltoniana*.

Inicialmente consideram-se sistemas dinâmicos conservativos em tempo contínuo, definidos em variedades Riemannianas, suaves, fechadas e conexas. Neste contexto é provada a C^1-*conjectura da estabilidade estrutural*, assim como resultados que relacionam a *hiperbolicidade uniforme* com as propriedades de *sombreamento* e de *expansividade*. Por fim, é descrito um *cenário geral* para a dinâmica contínua conservativa de sistemas definidos em variedades com dimensão superior a 3.

Um C^1-campo vectorial com divergência nula satisfaz a *propriedade estrela* se qualquer campo vectorial com divergência nula numa C^1-vizinhança do campo inicial tem todas as singularidades e todas as órbitas fechadas hiperbólicas. Nesta tese prova-se que todo o C^1-campo vectorial com divergência nula com a *propriedade estrela* é *uniformemente hiperbólico* e, em particular, não possui singularidades. Segundo este resultado, provar a hiperbolicidade uniforme para C^1-campos vectoriais com divergência nula equivale a provar que o campo satisfaz a propriedade estrela. Este resultado é posteriormente utilizado para provar que um C^1-campo vectorial com divergência nula e *estruturalmente estável* é, de facto, *uniformemente hiperbólico*.

Posteriormente, prova-se a equivalência entre as seguintes quatro propriedades:

- um C^1-campo vectorial com divergência nula está no C^1-interior do conjunto dos campos vectoriais *expansivos* com divergência nula;

- um C^1-campo vectorial com divergência nula está no C^1-interior do conjunto dos campos vectoriais com divergência nula que verificam a propriedade de *sombreamento*;

- um C^1-campo vectorial com divergência nula está no C^1-interior do conjunto dos

campos vectoriais com divergência nula que verificam a propriedade de *sombreamento Lipschitz*;

- um C^1-campo vectorial com divergência nula é *uniformemente hiperbólico*.

O ingrediente chave nestas provas é a caracterização dos campos vectoriais com divergência nula com a propriedade estrela como sendo uniformemente hiperbólicos.

Em [18], Bessa e Rocha descrevem um cenário geral para a dinâmica conservativa em dimensão 3. Nesta tese generaliza-se este resultado para sistemas definidos em variedades com dimensão superior a 3. Prova-se que um C^1-campo vectorial com divergência nula nestas condições pode ser C^1-aproximado por um campo vectorial com divergência nula *uniformemente hiperbólico* ou então por um C^1-campo vectorial com divergência nula com *ciclos heterodimensionais*.

A última parte desta tese reúne resultados de dinâmica Hamiltoniana. Seja H um *Hamiltoniano* definido numa variedade simpléctica M, $e \in H(M) \subset \mathbb{R}$ e $\mathcal{E}_{H,e}$ uma componente conexa sem singularidades de $H^{-1}(\{e\})$. Um *sistema Hamiltoniano*, seja um tripleto $(H, e, \mathcal{E}_{H,e})$, é uniformemente hiperbólico se a componente $\mathcal{E}_{H,e}$ é uniformemente hiperbólica. Por outro lado, um sistema Hamiltoniano $(H, e, \mathcal{E}_{H,e})$ é um *sistema Hamiltoniano estrela* se todas as órbitas fechadas em $\mathcal{E}_{H,e}$ são uniformemente hiperbólicas e o mesmo vale para uma componente conexa de $\tilde{H}^{-1}(\{\tilde{e}\})$, perto de $\mathcal{E}_{H,e}$, para qualquer $\tilde{H}$ numa C^2-vizinhança de H e para qualquer $\tilde{e}$ numa vizinhança de e. Neste contexto, prova-se que um *sistema Hamiltoniano estrela* definido numa variedade simpléctica de dimensão 4 é *uniformemente hiperbólico*. Prova-se ainda a *conjectura da estabilidade estrutural* para sistemas Hamiltonianos em variedades de dimensão 4.

Por fim, mostra-se que, dado um Hamiltoniano genérico H, existe um conjunto aberto e denso $\mathcal{S}(H)$ em $H(M)$ tal que, para qualquer $e \in \mathcal{S}(H)$, toda a componente conexa $\mathcal{E}_{H,e} \subset H^{-1}(\{e\})$ é *topologicamente misturadora*. O resultado essencial para concluir esta prova é uma versão do *lema da conexão de pseudo-órbitas para Hamiltonianos*. Nesta tese é apresentado o enunciado do lema utilizado, assim como uma ideia da sua prova. Este resultado genérico é relevante, na medida em que permite obter a prova de resultados como a dicotomia de Newhouse para Hamiltonianos, entre outros. Contudo, estas aplicações são direccionadas para um trabalho futuro.

ABSTRACT

This thesis contains results on *conservative* and on *Hamiltonian dynamics*.

Here, we include the proof of the C^1*-structural stability conjecture*, as well as results relating uniform hyperbolicity, *shadowing* and *expansiveness* properties for C^1-divergence-free vector fields defined on a closed, connected and smooth Riemannian manifold with dimension greater than 2. When the dimension of the manifold is greater than 3, we also describe a *general scenario* for this kind of dynamics.

A C^1-divergence-free vector field satisfies the *star property* if any divergence-free vector field in some C^1-neighborhood has all singularities and all closed orbits hyperbolic. We prove that any divergence-free vector field satisfying the *star property* is *uniformly hyperbolic*. This result is relevant because, from it, to prove uniform hyperbolicity for divergence-free vector fields it is enough to show that the vector field satisfies the star property. Afterwards, this result is used to prove that a C^1*-structurally stable* divergence-free vector field is, in fact, a *uniformly hyperbolic* divergence-free vector field, beyond other results.

Later, we prove that the following properties are equivalent:

- a C^1-divergence-free vector field is in the C^1-interior of the set of *expansive* divergence-free vector fields;

- a C^1-divergence-free vector field is in the C^1-interior of the set of divergence-free vector fields which satisfy the *shadowing property*;

- a C^1-divergence-free vector field is in the C^1-interior of the set of divergence-free vector fields which satisfy the *Lipschitz shadowing property*;

- a C^1-divergence-free vector field is *uniformly hyperbolic*.

Again, a cornerstone to prove this result is the equality between star and uniformly hyperbolic C^1-divergence-free vector-fields, obtained before.

In [18], Bessa and Rocha describe a general scenario for the conservative dynamics in dimension 3. In this thesis, we generalize this result for manifold with dimension greater that 3, by proving that any divergence-free vector field can be C^1-approximated by a *uniformly hyperbolic* divergence-free vector field, or else by a divergence-free vector field *exhibiting a heterodimensional cycle*.

Now, let H be a *Hamiltonian* defined on a symplectic manifold M, $e \in H(M) \subset \mathbb{R}$ and $\mathcal{E}_{H,e}$ a connected component of $H^{-1}(\{e\})$ without singularities. A *Hamiltonian system*, say a triplet $(H, e, \mathcal{E}_{H,e})$, is *uniformly hyperbolic* if $\mathcal{E}_{H,e}$ is uniformly hyperbolic. A Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is a *Hamiltonian star system* if all the closed orbits of $\mathcal{E}_{H,e}$ are hyperbolic and the same holds for a connected component of $\tilde{H}^{-1}(\{\tilde{e}\})$, close to $\mathcal{E}_{H,e}$, for any $\tilde{H}$ in some C^2-neighborhood of H and for any $\tilde{e}$ in some neighborhood of e. In this context, we show that a *Hamiltonian star system* defined on a 4-dimensional symplectic manifold is *uniformly hyperbolic*. Moreover, we prove the *structural stability conjecture* for Hamiltonian systems defined on a 4-dimensional symplectic manifold.

In the last part of this thesis, we show that, given a C^2-generic Hamiltonian H, there exists an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that, for any $e \in \mathcal{S}(H)$, every $\mathcal{E}_{H,e} \subset H^{-1}(\{e\})$ is *topologically mixing*. The most important ingredient to show this result is a version of the *connecting lemma for pseudo-orbits of Hamiltonians*, whose highlights of the proof are also stated. This theorem is relevant, because it allows us to show results as the *Newhouse Dichotomy for Hamiltonians*, among others. But these applications are postponed to a future work.

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ONE

INTRODUCTION AND RESULTS' STATEMENT

This thesis is a contribution to issues concerning on the *structural stability conjecture*, on the *shadowing and expansiveness properties* of a dynamical system, on the description of a *general scenario for dynamics* and on the *generic transitivity*. These questions will be addressed from the standpoint of conservative and Hamiltonian dynamics.

This chapter brings together the main notation and assumptions in order to properly state the main results.

1.1 Structural stability conjecture

One of the most challenging problems in the modern theory of dynamical systems, posed by Palis and Smale in 1970, is the well-known *structural stability conjecture* (see [61]).

Conjecture 1.1 *A C^r-structurally stable system satisfies the Axiom A and the strong transversality conditions, for $r \geq 1$.*

Let S be a system defined on a closed manifold. The notion of structural stability was firstly introduced in the mid 1930's by Andronov and Pontrjagin (see [4]) and this concept is intrinsically related to *uniform hyperbolicity*.

Roughly speaking, a system is *uniformly hyperbolic* if the tangent bundle splits into two invariant sub-bundles, one where the action is uniformly contracting and other where the action is uniformly expanding, and, in the continuous-time case, a one dimensional fiber including the direction of the flow. A system S is C^r*-structurally stable* ($r \geq 1$) if there exists a C^r-neighborhood $\mathcal{U}$ of S such that any other system in $\mathcal{U}$ is topologically conjugated to S.

We say that the system S satisfies the *Axiom A* property if the closure of its closed orbits is equal to the non-wandering set, $\Omega(S)$, and, moreover, this set is hyperbolic. Notice that a conservative system satisfying the Axiom A property is actually uniformly hyperbolic, since its non-wandering set coincides with the entire manifold. By the spectral decomposition of an Axiom A system S, we have that $\Omega(S) = \cup_{i=1}^{k}\Lambda_i$, where each set Λ_i is called a basic piece. We define an order relation by $\Lambda_i \prec \Lambda_j$ if there exists x (outside $\Lambda_i \cup \Lambda_j$) such that $\alpha(x) \subset \Lambda_i$ and $\omega(x) \subset \Lambda_j$. The system S has a *cycle* if there exists a cycle with respect to $\prec$ (see [72], for more details).

A cornerstone on the structural stability conjecture was the remarkable proof for C^1-diffeomorphisms, achieved by Mañé, in [50]. In fact, in the early 1980's, Mañé started to define the set $\mathcal{F}^1$ as the set of diffeomorphisms having a C^1-neighborhood $\mathcal{U}$ such that every diffeomorphism inside $\mathcal{U}$ has all periodic orbits of hyperbolic type. A system in $\mathcal{F}^1$ is called a *star system* or a system satisfying the *star property*. It is known that Ω-stable diffeomorphisms belong to $\mathcal{F}^1$ and that if $f \in \mathcal{F}^1$ then $\Omega(f) = \overline{Per(f)}$ (see [35, 52]). Thus, the structural stability conjecture is contained in the following.

Conjecture 1.2 *The non-wandering set of a star system is hyperbolic.*

The set $\mathcal{F}^1$ is related to the structural stability since the proof that a C^1-structural stable system satisfies the Axiom A property mainly uses the fact that the system is in $\mathcal{F}^1$. We point out that classic results imply that being in $\mathcal{F}^1$ is a necessary condition to satisfy the Axiom A and the strong transversality conditions (see [50] and the references wherein).

In [51], Mañé proved Conjecture 1.2 for diffeomorphisms defined on surfaces: any surface diffeomorphism of $\mathcal{F}^1$ satisfies the Axiom A and the no-cycle conditions. Later, in [43], Hayashi extended this result for higher dimensions. In 1988, Mañé presented a proof of Conjecture 1.1 for C^1-diffeomorphisms (see [50]). We point out that, after the proof of the C^1-structural stability conjecture for diffeomorphisms, Hayashi proves this conjecture for C^1-flows, in [41, 42]. Later Gan gives a different proof of this conjecture for C^1-flows (see [36]). Recently, Bessa and Rocha presented, in [20], a proof of the C^1-structural stability conjecture for conservative flows defined on a 3-dimensional manifold.

Nevertheless, the C^r-structural stability conjecture remains wide open for higher topologies ($r \geq 2$). This is explained, in particular, because many of the C^1-perturbation arguments, as the Closing Lemma, the Connecting Lemma and the Franks Lemma, are either unknown or they are false in higher topologies (see further details in [40, 65, 68, 80]).

Even for the continuous-time case, the proof of Conjecture 1.1 is simplified if we firstly prove Conjecture 1.2. In this context, the set analogous to $\mathcal{F}^1$ is traditionally denoted by $\mathcal{G}^1$, in which the hyperbolicity of the flow equilibria is also imposed.

The first results on this thesis are about the proof of Conjecture 1.2 for conservative star flows defined on high-dimensional manifolds and also for 4-dimensional Hamiltonian systems. These results will be used later to prove Conjecture 1.1 for high-dimensional conservative flows and for 4-dimensional Hamiltonian flows. In order to properly state these results, let us introduce some definitions.

From now on, M^d, sometimes called M, denotes a d-dimensional, ($d \geq 2$), compact, boundary-less, connected and smooth Riemannian manifold, endowed with a volume form, which has associated a measure μ, called the Lebesgue measure. Also, denote by $dist$ the Riemannian distance and consider, for $\epsilon > 0$ and $p \in M$, the open balls $B_\epsilon(p) = \{x \in M : dist(x,p) < \epsilon\}$.

Denote by $\mathfrak{X}^r(M)$ the set of vector fields defined on M, endowed with the C^r Whitney topology ($r \geq 1$). If the divergence of a C^r-vector field X is zero then we call X a C^r-divergence-free vector field. Let $\mathfrak{X}^r_\mu(M)$ denote the set of divergence-free vector fields defined on M, endowed with the induced C^r Whitney topology. A C^r-vector field $X : M \to TM$ generates a *flow* $X^t : M \to M$, which is a smooth 1-parameter group for $t \in \mathbb{R}$, satisfying

$$\frac{d}{dt}X^t|_{t=s}(p) = X(X^s(p)) \text{ and } X^0 = id.$$

If X is a divergence-free vector field then X^t is called a *conservative* flow. The linear part of the flow X^t, called *tangent flow*, $DX^t_p : T_pM \longrightarrow T_{X^t(p)}M$, for $p \in M$, satisfies

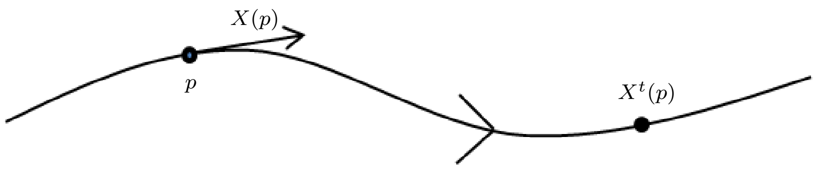

Figure 1.1: Representation of a flow.

the linearized differential equation

$$\frac{d}{dt}DX_p^t = (DX_{X^t(p)}) \circ DX_p^t,$$

where $DX_p : T_pM \longrightarrow T_pM$. Let $supp(X) = \overline{\{x \in M : X(x) \neq \vec{0}\}}$ denote the *support of* X. From now on, we are restricted to the C^1-topology ($r = 1$).

A *closed orbit* γ of X is a non-constant integral curve $\gamma : [a, b] \to M$ of X such that $\gamma(a) = \gamma(b)$. We define b as the smallest number greater than a satisfying $\gamma(a) = \gamma(b)$. Observe that the period of γ is $b - a$. For simplicity, sometimes we call $p \in \gamma$ a closed orbit. So, the set of *closed orbits* associated to the vector field X is denoted by

$$Per(X) = \{p \in M : \exists\, t > 0\,, X^t(p) = p\}.$$

Given a closed orbit γ and any $p \in \gamma$, if $\pi > 0$ is the least number such that $X^\pi(p) = p$ then γ is a closed orbit with *period* π.

Denote by $Per_\pi(X)$ the set of closed orbits with period less or equal than π of the vector field X and by $Per^\pi(X)$ the set of closed orbits with period greater than π of the vector field X. Obviously, $Per(X) = Per_\pi(X) \cup Per^\pi(X)$.

The set of *singularities* of the vector field X is denoted by

$$Sing(X) = \{p \in M : X(p) = \vec{0}\}.$$

Singularities and closed orbits of X are called *critical points* and are denoted by

$$Crit(X) = Sing(X) \cup Per(X).$$

If $p \notin Sing(X)$ then p is called a *regular* point and if $Sing(X) = \emptyset$ then M is said *regular*.

Before stating the definition of *star vector fields* for the continuous-time case, let us explain what does mean a singularity and a closed orbit to be hyperbolic.

Let γ be a closed orbit of X, take $p \in \gamma$ and denote by Σ a $(\dim(M)-1)$-*transversal section* to X at p. Poincaré defined a map f from $\tilde{\Sigma} \subset \Sigma$ to Σ, called *the Poincaré first return map* of the trajectories on Σ, such that, for any point $x \in \Sigma$ in a small neighborhood of p, the ω-trajectory of x will intersect Σ again at some point y at some time t close to the period of p.

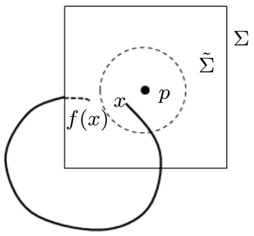

Figure 1.2: Representation of the Poincaré first return map.

A closed orbit γ of X is *hyperbolic* if $p \in \gamma$ is a hyperbolic fixed point of the Poincaré first return map. A singularity q of a C^1-vector field X is *hyperbolic* if the eigenvalues of DX_q are not purely imaginary. We say that any element of $Crit(X)$ is hyperbolic, if any singularity and any closed orbit of X is hyperbolic.

Definition 1.1 *A C^1-vector field X is a star vector field if there exists a C^1-neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^1(M)$ such that, for any $Y \in \mathcal{U}$, any element of the set $Crit(Y)$ is hyperbolic. Moreover, a vector field $X \in \mathfrak{X}^1_\mu(M)$ is a divergence-free star vector field if there exists a C^1-neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^1_\mu(M)$ such that, for any $Y \in \mathcal{U}$, any element of the set $Crit(Y)$ is hyperbolic. Note that if $X \in \mathfrak{X}^1_\mu(M)$ is a star vector field then X is a divergence-free star vector field. The set of C^1-star vector fields is denoted by $\mathcal{G}^1(M)$ and the set of C^1-divergence-free star vector fields is denoted by $\mathcal{G}^1_\mu(M)$.*

Observe that, in the previous definition, the hyperbolicity imposed at the critical points is not uniform. So, the hyperbolicity constants depend on the critical point.

By definition, $\mathcal{G}^1(M)$ and $\mathcal{G}^1_\mu(M)$ are C^1-open subsets of $\mathfrak{X}^1(M)$ and $\mathfrak{X}^1_\mu(M)$, respectively.

Given that Definition 1.1 concerns only to critical points and that the hyperbolicity on critical points is merely orbit-wise, the star property looks, a priori, quite a weak

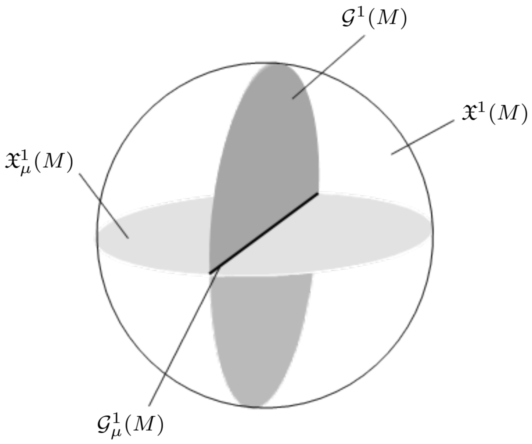

Figure 1.3: Representation of the sets $\mathcal{G}^1(M)$ and $\mathcal{G}^1_\mu(M)$.

property. However, as stated in Theorem 1 ahead, for the divergence-free setting, it is not.

Let us now state the usual definition of *uniformly hyperbolic set*.

Definition 1.2 *Given $X \in \mathfrak{X}^1(M)$, an X^t-invariant, compact and regular set Λ on M is uniformly hyperbolic if there exist a DX^t-invariant splitting $T_\Lambda M = E^s_\Lambda \oplus \mathbb{R}X(\Lambda) \oplus E^u_\Lambda$ and constants $c > 0$ and $0 < \kappa < 1$ such that, for any $x \in \Lambda$ and any $t > 0$, we have:*

$$\left\|DX^t_x|_{E^s_x}\right\| \leq c\kappa^t \text{ and } \left\|DX^{-t}_{X^t(x)}|_{E^u_{X^t(x)}}\right\| \leq c\kappa^t,$$

where $\mathbb{R}X(x)$ denotes the space spanned by $X^t(x)$.

Observe that the constants c and κ, in the previous definition, do not depend on $x \in \Lambda$.

The definition of *Anosov vector field* is related with the definition of uniformly hyperbolic set.

Definition 1.3 *A C^1-vector field X defined on M is called Anosov if the manifold M is uniformly hyperbolic. Let $\mathcal{A}^1(M)$ denote the set of Anosov C^1-vector fields and denote by $\mathcal{A}^1_\mu(M)$ the set of Anosov C^1-divergence-free vector fields defined on M.*

The sets $\mathcal{A}^1(M)$ and $\mathcal{A}^1_\mu(M)$ are C^1-*open* subsets of $\mathfrak{X}^1(M)$ and $\mathfrak{X}^1_\mu(M)$, respectively (see [5]).

Remark 1 *Note that, if X is an Anosov vector field then $Sing(X) = \emptyset$. In fact, if there is $q \in Sing(X)$ then q is hyperbolic, therefore isolated and satisfying $T_q M = E^s_q \oplus E^u_q$.*

This means that q is surrounded by regular hyperbolic points p satisfying

$$T_pM = E^s_p \oplus \mathbb{R}X(p) \oplus E^u_p.$$

But this is a contradiction, since the fibers of T_xM depend continuously on $x \in M$.

A star vector field may fail to have a hyperbolic non-wandering set, as the famous Lorenz attractor shows (see [39]), since the hyperbolic saddle-type singularity is accumulated by hyperbolic closed orbits, which are contained in the non-wandering set. This prevents the flow to be Axiom A. There are also examples of star vector fields that fail to have the critical elements dense in the non-wandering set (see [31]) or, even satisfying the Axiom A property, still may fail to satisfy the no-cycle condition (see [49]). However, all these star vector fields counterexamples exhibit singularities. Recently, Gan and Wen proved, in [37], that a star C^1-vector field defined on a d-dimensional manifold ($d \geq 3$) with no singularities is Axiom A without cycles. Later, based in lower-dimensional conservative-type seminal ideas of Mañé and on the openness of the set of Anosov divergence-free vector fields, Bessa and Rocha proved, in [20], that $\mathcal{G}^1_\mu(M^3) = \mathcal{A}^1_\mu(M^3)$. The proof of this result cannot be trivially adapted to higher dimensions. We remark that, in dimension 3, divergence-free vector fields with a dominated splitting are, in fact, Anosov. This happens because the normal bundle is splitted in two 1-dimensional subbundles (see [20, Lemma 3.2]). However, this is not necessarily true in higher dimensions.

The first theorem is the high-dimensional version of this later result and it is used to derive the proof of Conjecture 1.2 in the C^1-divergence-free vector fields context.

Theorem 1 *([34, Theorem 1]) If $X \in \mathcal{G}^1_\mu(M^d)$ then $Sing(X) = \emptyset$ and $X \in \mathcal{A}^1_\mu(M^d)$, for $d \geq 4$.*

The main novelties in the proof of Theorem 1 are the use of a new strategy to prove the absence of singularities and the adaption of an argument of Mañé in [51] to show hyperbolicity from a dominated splitting, which follows easily when we are in dimension 3.

So, from the 3-dimensional result due to Bessa and Rocha and from Theorem 1, we have that $\mathcal{G}^1_\mu(M^d) = \mathcal{A}^1_\mu(M^d)$, for $d \geq 3$.

The structural stability conjecture can also be stated in the Hamiltonian context. For that, we need to use specific tools and several recent results on Hamiltonian dynamics. It is worth pointing out that part of the difficulty of this problem consists in transposing in a proper way concepts from the general vector field setting to the Hamiltonian one.

Let (M^{2d}, ω) be a symplectic manifold, where M^{2d} $(d \geq 2)$ is an even-dimensional, compact, boundary-less, connected and smooth Riemannian manifold, endowed with a symplectic form ω. Denote by $C^s(M, \mathbb{R})$ the set of C^s-real-valued functions on M and call $H \in C^s(M, \mathbb{R})$ a *C^s-Hamiltonian*, for $s \geq 2$. From now on, we set $s = 2$.

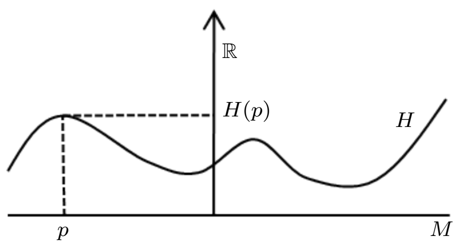

Figure 1.4: Representation of a Hamiltonian function H.

Given a Hamiltonian H, we can define the *Hamiltonian vector field* X_H by

$$\omega(X_H(p), u) = d_pH(u),\ \forall u \in T_pM,$$

which generates the Hamiltonian flow X_H^t.

Remark 2 *Observe that H is C^2 if and only if X_H is C^1 and that, since H is continuous and M is compact and boundary-less, $Sing(X_H) \neq \emptyset$.*

A scalar $e \in H(M) \subset \mathbb{R}$ is called an *energy* of H. An *energy hypersurface* $\mathcal{E}_{H,e}$ is a connected component of $H^{-1}(\{e\})$, called *energy level set*.

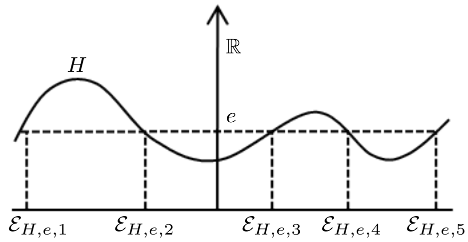

Figure 1.5: Representation of energy hypersurfaces.

The energy level set $H^{-1}(\{e\})$ is said *regular* if any energy hypersurface of $H^{-1}(\{e\})$ is regular. In this case, we can also say that the energy e is regular. Observe that a regular *energy hypersurface* is a X_H^t-invariant, compact and $(2d-1)$-dimensional manifold.

Definition 1.4 *Consider a Hamiltonian $H \in C^2(M, \mathbb{R})$, an energy $e \in H(M)$ and a regular energy hypersurface $\mathcal{E}_{H,e}$. The triplet $(H, e, \mathcal{E}_{H,e})$ is called Hamiltonian system and the pair (H, e) is called Hamiltonian level.*

A *Hamiltonian level* (H, e) is said *regular* if the energy level set $H^{-1}(\{e\})$ is regular. If (H, e) is regular then $H^{-1}(\{e\})$ corresponds to the union of a finite number of closed connected components, that is, $H^{-1}(\{e\}) = \sqcup_{i=1}^{I_e} \mathcal{E}_{H,e,i}$, for $I_e \in \mathbb{N}$.

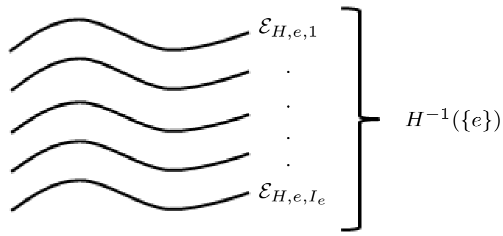

Figure 1.6: Representation of a regular energy level.

Fixing a small neighborhood $\mathcal{W}$ of a regular energy hypersurface $\mathcal{E}_{H,e}$, there exist a small neighborhood $\mathcal{U}$ of the Hamiltonian H and $\epsilon > 0$ such that, for any $\tilde{H} \in \mathcal{U}$ and for any $\tilde{e} \in (e-\epsilon, e+\epsilon)$, we have $\tilde{H}^{-1}(\{\tilde{e}\}) \cap \mathcal{W} = \mathcal{E}_{\tilde{H},\tilde{e}}$. The energy hypersurface $\mathcal{E}_{\tilde{H},\tilde{e}}$ is called *analytic continuation* of $\mathcal{E}_{H,e}$.

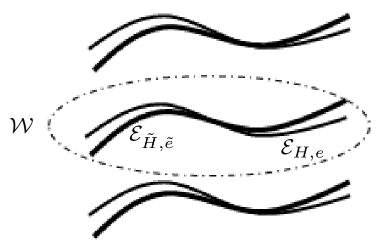

Figure 1.7: Representation of a analytic continuation of $\mathcal{E}_{H,e}$.

Accordingly with the previous notions, we introduce the definition of *Hamiltonian star system*.

Definition 1.5 *A Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is called a Hamiltonian star system if there exist a neighborhood $\mathcal{U}$ of H and $\epsilon > 0$ such that, for any $\tilde{H} \in \mathcal{U}$ and any*

$\tilde{e} \in (e-\epsilon, e+\epsilon)$, all the closed orbits of $\mathcal{E}_{\tilde{H},\tilde{e}}$ are hyperbolic. We denote by $\mathcal{E}^{\star}_{H,e}$ the regular energy hypersurface with the previous property and by $\mathcal{G}^2_\omega(M^{2d})$ the set of triplets of all Hamiltonian star systems defined on a $2d$-dimensional symplectic manifold, for $d \geq 2$.

Note that a Hamiltonian H can appear several times in the triplets in $\mathcal{G}^2_\omega(M^{2d})$. This is possible if H is followed by a different energy e or, even with the same energy, if it is grouped with a different energy hypersurface.

The next definition states when a Hamiltonian system is *Anosov*.

Definition 1.6 *A Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is Anosov if $\mathcal{E}_{H,e}$ is uniformly hyperbolic for the Hamiltonian flow X^t_H associated to H. Let $\mathcal{A}^2_\omega(M^{2d})$ denote the set of triplets of Anosov Hamiltonian systems, defined on a $2d$-dimensional symplectic manifold, for $d \geq 2$.*

To prove Conjecture 1.1 in the Hamiltonian context, we need to prove that the set of Anosov Hamiltonian systems is open and that its elements are structurally stable. For such, let us state the definition of an *open set of Hamiltonian systems* and of a *structurally stable Hamiltonian system*.

Definition 1.7 *Let $\mathcal{H}$ be a set of Hamiltonian systems. The set $\mathcal{H}$ is open if, for any Hamiltonian system $(H, e, \mathcal{E}_{H,e}) \in \mathcal{H}$, there exist a small neighborhood $\mathcal{U}$ of H and $\epsilon > 0$ such that, for any $\tilde{H} \in \mathcal{U}$ and any $\tilde{e} \in (e-\epsilon, e+\epsilon)$, the Hamiltonian system $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}})$ belongs to $\mathcal{H}$.*

Note that the neighborhood of $(H, e, \mathcal{E}_{H,e}) \in \mathcal{H}$ is determined by $\mathcal{U}$ and ϵ.

The following result, refers to the *openness of Anosov Hamiltonian systems* defined on a $2d$-dimensional symplectic manifold ($d \geq 2$).

Theorem 2 *([13, Theorem 3]) The set $\mathcal{A}^2_\omega(M^{2d})$ is open, for $d \geq 2$.*

In the next definition, we define a structurally stable Hamiltonian system.

Definition 1.8 *Consider a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$. If there exist a small C^2-neighborhood $\mathcal{U}$ of H and $\epsilon > 0$ such that, for any $\tilde{H} \in \mathcal{U}$ and any $\tilde{e} \in (e-\epsilon, e+\epsilon)$ there exists a homeomorphism between $\mathcal{E}_{H,e}$ and $\mathcal{E}_{\tilde{H},\tilde{e}}$, preserving orbits and their orientations, we say that the Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is C^2-structurally stable.*

From this definition, we have the following result.

Theorem 3 *([13, Theorem 3]) The elements of $\mathcal{A}^2_\omega(M^{2d})$ are C^2-structurally stable, for $d \geq 2$.*

Now, we are in conditions to state the version of Conjecture 1.2 for Hamiltonians.

Theorem 4 *([13, Theorem 1]) If $(H, e, \mathcal{E}^\star_{H,e}) \in \mathcal{G}^2_\omega(M^4)$ then $(H, e, \mathcal{E}^\star_{H,e}) \in \mathcal{A}^2_\omega(M^4)$.*

The previous theorem states that a Hamiltonian star system, defined on a 4-dimensional symplectic manifold, is, in fact, an Anosov Hamiltonian system. To prove this, we follow the strategy described by Bessa and Rocha, in [20], for conservative flows. This result is only obtained in dimension 4 because its proof makes use of some results that are only available in low dimension.

From Theorem 1 and Theorem 4, we can derive some interesting results, as an answer to the structural stability conjecture. Let us start with the definition of *structurally stable* vector field.

Definition 1.9 *A C^1-vector field X is called C^1-structurally stable if there exists a C^1-neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^1(M)$ such that, for any $Y \in \mathcal{U}$, there exists a homeomorphism between X^t and Y^t, preserving orbits and their orientations. Denote by $\mathcal{SS}^1(M)$ the set of C^1-structurally stable vector fields and by $\mathcal{SS}^1_\mu(M)$ the set of C^1-structurally stable divergence-free vector fields.*

It is also well-known that Anosov C^1-vector fields are C^1-*structurally stable* (see [5]). Hence, Conjecture 1.1 states the equivalence between uniform hyperbolicity and C^1-structural stability.

In this thesis, we generalize the result [20, Theorem 1.3] to higher dimensions.

Theorem 5 *([34, Theorem 2]) If $X \in \mathcal{SS}^1_\mu(M^d)$ then $X \in \mathcal{A}^1_\mu(M^d)$, for $d \geq 4$.*

The following result is a 4-dimensional proof of the structural stability conjecture for Hamiltonians. It says that a C^2-structurally stable Hamiltonian system, defined on a 4-dimensional symplectic manifold, is Anosov.

Theorem 6 *([13, Theorem 2]) If $(H, e, \mathcal{E}_{H,e})$ is a C^2-structurally stable Hamiltonian system then $(H, e, \mathcal{E}_{H,e}) \in \mathcal{A}^2_\omega(M^4)$.*

Now, we want to state some other consequences of Theorem 1 and Theorem 4. For such, we introduce some extra definitions.

Definition 1.10 *Let $\mathcal{V}$ be an open subset of $\mathfrak{X}^1_\mu(M)$. We say that a C^1-vector field X is isolated in the boundary of the set $\mathcal{V}$ if $X \notin \mathcal{V}$ and, given a small neighborhood $\mathcal{U}$ of X, any vector field $Y \in \mathcal{U}\backslash X$ belongs to $\mathcal{V}$.*

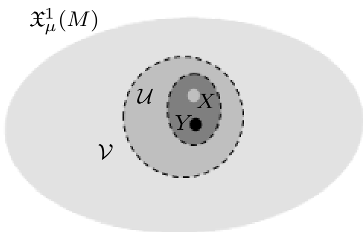

Figure 1.8: Vector field X isolated in the boundary of a set $\mathcal{V}$.

Accordingly with this definition, by Theorem 1, we obtain the following result.

Corollary 1 *([34, Corollary 1]) The boundary of the set $\mathcal{A}^1_\mu(M^d)$ has no isolated points, for $d \geq 4$.*

We can also try to describe the *boundary of a Hamiltonian system*.

Definition 1.11 *Let $\mathcal{H}$ be a set of Hamiltonian systems. We say that a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is isolated in the boundary of $\mathcal{H}$ if $(H, e, \mathcal{E}_{H,e}) \notin \mathcal{H}$ but, given any small C^2-neighborhood $\mathcal{U}$ of H and $\delta > 0$, for any $\tilde{H} \in \mathcal{U}\backslash H$ and for any $\tilde{e} \in (e-\delta, e+\delta)\backslash\{e\}$, we have that the Hamiltonian system $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}})$ belongs to $\mathcal{H}$.*

Now, following Theorem 4, we can derive an analogous result to Corollary 1, but for 4-dimensional symplectic manifolds.

Corollary 2 *([13, Corollary 1]) The boundary of the set $\mathcal{A}^2_\omega(M^4)$ has no isolated points.*

Now, we want to state a corollary of Theorem 1, concerning on Kupka-Smale vector fields.

Definition 1.12 *A vector field $X \in \mathfrak{X}^1(M)$ is Kupka-Smale if all the elements of the set $Crit(X)$ are hyperbolic and their stable and unstable manifolds intersect transversely. Denote by $\mathcal{KS}^1(M)$ the set of C^1-Kupka-Smale vector fields and by $\mathcal{KS}^1_\mu(M)$ the set of C^1-Kupka-Smale divergence-free vector fields.*

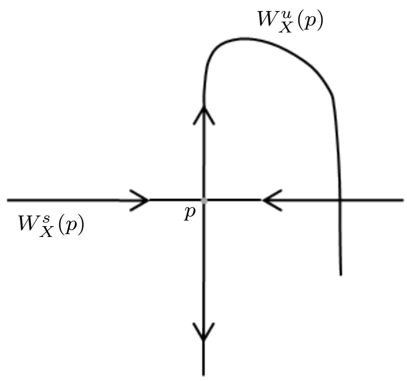

Figure 1.9: Representation of a critical point p of a Kupka-Smale vector field.

See Section 2.1.3, for more details on the invariant manifolds of a hyperbolic set.

In [73], Smale shows that the set $\mathcal{KS}^1(M)$ is a C^1-residual subset of $\mathfrak{X}^1(M)$. Later, Robinson proved this property for divergence-free vector fields. So, the set $\mathcal{KS}^1_\mu(M)$ is a C^1-residual subset of $\mathfrak{X}^1_\mu(M)$ (see [69]). From [20, Theorem 1.2] and Theorem 1, it is straightforward to obtain the following result.

Corollary 3 *If $X \in int(\mathcal{KS}^1_\mu(M^d))$ then $X \in \mathcal{A}^1_\mu(M^d)$, for $d \geq 3$.*

We remark that $int(S)$ stands for the C^1-interior of the set $S \subset \mathfrak{X}^1_\mu(M)$. This means that Theorem 1 gives an immediate proof, for divergence-free vector fields, of the result shown by Toyoshiba, in [75].

In this section, we have emphasized the implication of Theorem 1 and Theorem 4 in the proof of the structural stability conjecture for high-dimensional divergence-free vector fields and for 4-dimensional Hamiltonian systems. It was also stated that these theorems lead to some other results.

1.2 Shadowing and expansiveness

The theory of shadowing studies the closeness of pseudo-orbits and exact trajectories of dynamical systems. A dynamical system has some shadowing property if any pseudo-orbit with small error is, in some sense, close to some exact trajectory. The notions of *pseudo-orbit* and *being close* can be formalized in several ways. Therefore, since Anosov and Bowen various types of shadowing properties have been introduced in several contexts.

We want to state the definition of *shadowing* for continuous-time systems. First, define Rep as the set of the increasing homeomorphisms $\alpha : \mathbb{R} \to \mathbb{R}$, called *reparametrizations*, satisfying $\alpha(0) = 0$. Fixing $\epsilon > 0$, define the set

$$Rep(\epsilon) = \Big\{\alpha \in Rep : \Big|\frac{\alpha(t)}{t} - 1\Big| < \epsilon, t \in \mathbb{R}\backslash\{0\}\Big\}.$$

When we choose a reparametrization α in the previous set, we want $\alpha(t)$ to be taken arbitrarily close to the identity.

Definition 1.13 *Fix $T > 0$ and $\delta > 0$. A map $\psi : \mathbb{R} \to M$ is a (δ, T)-pseudo-orbit of a flow X^t if $dist(X^t(\psi(\tau)), \psi(\tau + t)) < \delta$, for any $\tau \in \mathbb{R}$ and any $|t| \leq T$. A pseudo-orbit ψ of a flow X^t is said to be ϵ-shadowed by some orbit of X^t if there is $x \in M$ and a reparametrization $\alpha \in Rep(\epsilon)$ such that $dist(X^{\alpha(t)}(x), \psi(t)) < \epsilon$, for every $t \in \mathbb{R}$.*

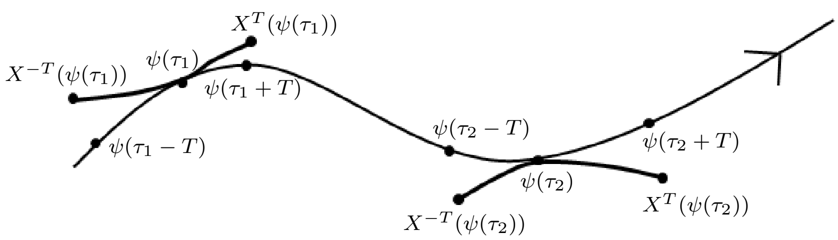

Figure 1.10: Representation of a pseudo-orbit.

Note that ψ is not assumed to be continuous.

Now, we are ready to properly state the definition of shadowing for C^1-vector fields, in which we need a reparameterization of shadowing orbits.

Definition 1.14 *A C^1-vector field X satisfies the shadowing property if, for any $\epsilon > 0$ and any $T > 0$, there is $\delta > 0$ such that any (δ, T)-pseudo-orbit ψ is ϵ-shadowed by some orbit of X. Let $\mathcal{S}^1(M)$ and $\mathcal{S}^1_\mu(M)$ denote the sets of vector fields in $\mathfrak{X}^1(M)$ and $\mathfrak{X}^1_\mu(M)$, respectively, satisfying the shadowing property.*

Smale proved that a diffeomorphism in the C^1-interior of the set of diffeomorphisms with the shadowing property is C^1-structurally stable (see [73]). More recently, Lee and Sakai proved, in [47], that if X belongs to the interior of the set $\mathcal{S}^1(M)$ and has no singularities then X satisfies the Axiom A and the strong transversality conditions. For divergence-free vector fields, we prove the following result.

Theorem 7 *([33, Theorem 1]) If $X \in int(\mathcal{S}^1_\mu(M^d))$ then $X \in \mathcal{A}^1_\mu(M^d)$, for $d \geq 3$.*

The *Lipschitz shadowing property* is a stronger definition of shadowing.

Definition 1.15 *A C^1-vector field X satisfies the Lipschitz shadowing property if there are positive constants ℓ and δ_0 such that any (δ, T)-pseudo-orbit ψ, with $T > 0$ and $\delta \leq \delta_0$, is $\ell\delta$-shadowed by an orbit of X. Let $\mathcal{LS}^1(M)$ and $\mathcal{LS}^1_\mu(M)$ denote the sets of vector fields in $\mathfrak{X}^1(M)$ and $\mathfrak{X}^1_\mu(M)$, respectively, satisfying the Lipschitz shadowing property.*

By definition, it is immediate that the set $\mathcal{LS}^1(M)$ is a subset of $\mathcal{S}^1(M)$ and that the set $\mathcal{LS}^1_\mu(M)$ is a subset of $\mathcal{S}^1_\mu(M)$. Therefore, from Theorem 7, we have that the C^1-interior of the set $\mathcal{LS}^1_\mu(M)$ is contained in the set $\mathcal{A}^1_\mu(M)$.

In [74], Tikhomirov proved that a vector field in the C^1-interior of the set of vector fields with the Lipschitz shadowing property is structurally stable. Recently, Pilyugin and Tikhomirov proved that a C^1-diffeomorphism having the Lipschitz shadowing property is structurally stable (see [64]).

The following definition is the notion of *expansive vector field*, introduced by Bowen and Walters, in [28].

Definition 1.16 *A C^1-vector field X is expansive if, for any $\epsilon > 0$, there is $\delta > 0$ such that if $x, y \in M$ satisfy $dist(X^t(x), X^{\alpha(t)}(y)) \leq \delta$, for any $t \in \mathbb{R}$ and for some continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $y = X^s(x)$, where $|s| \leq \epsilon$. Denote by*

$\mathcal{E}^1(M) \subset \mathfrak{X}^1(M)$ the set of expansive vector fields and by $\mathcal{E}^1_\mu(M) \subset \mathfrak{X}^1_\mu(M)$ the set of expansive divergence-free vector fields, both endowed with the C^1 Whitney topology.

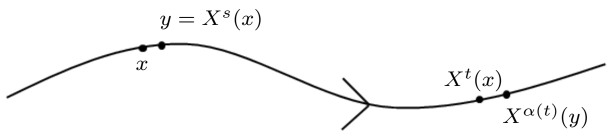

Figure 1.11: Representation of an expansive vector field's orbit.

This definition asserts that any two points whose orbits remain indistinguishable, up to any continuous time displacement, must be in the same orbit.

Observe that the reparametrization α, in Definition 1.16, is not assumed to be close to identity and that the expansiveness property does not depend on the choice of the metric on M.

In 1970's, Mañé proved that a diffeomorphism f in the C^1-interior of the set of expansive diffeomorphisms is Axiom A and satisfies the quasi-transversality condition (see [53]). Later, Moriyasu, Sakai and Sun proved the same result for vector fields, in [57]. Moreover, the authors proved that if $X \in int(\mathcal{E}^1(M))$ and has the shadowing property then X is Anosov. Recently, Pilyugin and Tikhomirov proved that an expansive diffeomorphism having the Lipschitz shadowing property is Anosov (see [64]). In the next result, we prove that a divergence-free vector field in the C^1-interior of the set of expansive divergence-free vector fields is actually Anosov.

Theorem 8 *([33, Theorem 1]) If $X \in int(\mathcal{E}^1_\mu(M^d))$ then $X \in \mathcal{A}^1_\mu(M^d)$, for $d \geq 3$.*

The expansiveness and the shadowing properties play an essential role in the investigation of the stability theory and the ergodic theory of Axiom A diffeomorphisms (see [26]). It is well-known that Anosov systems are expansive and satisfy the shadowing and the Lipschitz shadowing properties (see [5, 63]).

To conclude this section, we notice that, by Theorem 1, Theorem 7, Theorem 8 and Corollary 3, we have the following result.

Corollary 1.1 *For the conservative setting,*

$$\mathcal{G}^1_\mu(M) = \mathcal{A}^1_\mu(M) = int(\mathcal{S}^1_\mu(M)) = int(\mathcal{KS}^1_\mu(M)) = int(\mathcal{LS}^1_\mu(M)) = int(\mathcal{E}^1_\mu(M)).$$

1.3 General scenario for dynamics

At the second half of the 1960's, it was already clear that the set of uniformly hyperbolic systems is open but not dense. Thus, it triggered the beginning of the search for an answer to the following question.

Question 1.1 *Is it possible to look for a general scenario for dynamics?*

This search draws the attention to homoclinic orbits, that is, orbits that in the past and in the future converge to the same periodic orbit, which have been firstly considered by Poincaré, almost a century before. The creation or destruction of such orbits is, roughly speaking, what its meant by homoclinic bifurcations (see, for example, [62]). Based on these and other subsequent developments, in [60], Palis formulated Conjecture 1.3, concerning on hyperbolicity, homoclinic tangencies and heterodimensional cycles. Roughly speaking, a *homoclinic tangency* is a non-transverse intersection between the stable and unstable manifolds of a hyperbolic closed orbit of saddle-type. A heterodimensional cycle is a cyclical intersection between the invariant manifolds of two distinct hyperbolic critical points of saddle-type with different dimension of the unstable bundles (see Definition 2.6, in Section 2.1.3, for more details).

Conjecture 1.3 *Diffeomorphisms with either a homoclinic tangency or a heterodimensional cycle are C^r-dense in the complement of the C^r closure of hyperbolic diffeomorphisms ($r \geq 1$).*

In [67], Pujals and Sambarino proved this conjecture in the case of C^1-diffeomorphisms defined on a compact surface. Recently, Bessa and Rocha proved this conjecture for C^1-volume-preserving diffeomorphisms in [16]. In fact, the authors show that a volume-preserving diffeomorphism can be C^1-approximated by an Anosov volume-preserving diffeomorphism, or else by a volume-preserving diffeomorphism displaying a heterodimensional cycle. The authors also proved a similar result for symplectomorphisms.

For the continuous-time case, Arroyo and Hertz proved, in [9], an analogous statement of Conjecture 1.3 for C^1-vector fields defined on a 3-dimensional, compact manifold. In this context, besides *homoclinic tangencies* and heterodimensional cycles, the *singular cycles* are another homoclinic phenomenon that must be considered. The authors show that a vector field $X \in \mathfrak{X}^1(M^3)$ can be C^1-approximated by an Anosov vector field, or else by a vector field displaying a homoclinic tangency, or else by a vector field displaying a singular cycle. For the divergence-free context, Bessa and Rocha show, in [18], that any vector field X in $\mathfrak{X}^1_\mu(M^3)$ can be C^1-approximated by a divergence-free vector field which is Anosov, or else has a homoclinic tangency. In this paper, the authors left open the following question, related with Conjecture 1.3.

Question 1.2 *Can any vector field X in $\mathfrak{X}^1_\mu(M^d)$ be C^1-approximated by a divergence-free vector field exhibiting some form of hyperbolicity on M^d ($d \geq 4$), or by one exhibiting homoclinic tangencies, or else by one having a heterodimensional cycle?*

The following result is the answer to this question.

Theorem 9 *([34, Theorem 3]) If $X \in \mathfrak{X}^1_\mu(M^d)$, for $d \geq 4$, then X can be C^1-approximated by an Anosov divergence-free vector field, or else by a divergence-free vector field exhibiting a heterodimensional cycle.*

1.4 Topological transitivity

The topological transitivity is a global property of a dynamical system. As a motivation for this notion, we may think of a real physical system, where a state is never measured exactly. Thus, instead of points, we should study (small) open subsets of the phase space and describe how they move in that space. If each one of these open subsets meet each other by the action of the system after some time, then we say that the system is *topologically transitive*. Equivalently, if we take a compact phase space, we may say that the system has a *dense orbit*. However, if the open subsets remain inseparable after some time, by the iteration of the system, then we say that the system is *topologically mixing*. Obviously, a topologically mixing system is also a topologically transitive system.

The concept of transitivity goes back to Birkhoff. According to [38], Birkhoff used it in [21, 22]. Throughout in this thesis *transitive* will always mean topologically transitive.

There exists a lot of transitive systems, as the *irrational rotations* of $\mathbb{S}^1$, the *shift maps* and the *basic sets*. It is also well-known that $C^{1+\alpha}$-*Anosov systems* are ergodic and so transitive (see [5]). Nevertheless, transitivity is not an open property.

Question 1.3 *Can the transitivity property be generic?*

Some authors have been working on this question. The first remarkable result on this subject is due to Bonatti and Crovisier, in [24]. The authors show that, C^1-generically, a C^1-conservative diffeomorphism is transitive. Later, jointly with Arnaud, the authors extend this result for C^1-symplectic diffeomorphisms defined on a symplectic manifold (see [8]). Adapting the techniques used to prove these results to the continuous-time case, Bessa proved an analogous result for C^1-divergence-free vector fields. In fact, by a result due to Abdenur, Avila and Bochi (see [1]), Bessa was able to show that, C^1-generically, a divergence-free vector field is topologically mixing (see [11]).

Our contribution to this issue is the statement and the proof of a result that is an answer to Question 1.3 for Hamiltonian systems. Let us start with some definitions.

Definition 1.17 *A compact energy hypersurface $\mathcal{E}_{H,e}$ is topologically mixing if, for any open and non-empty subsets of $\mathcal{E}_{H,e}$, say U and V, there is $\tau \in \mathbb{R}$ such that $X_H^t(U) \cap V \neq \emptyset$, for any $t \geq \tau$. A regular Hamiltonian level (H,e) is topologically mixing if each one of the energy hypersurfaces of $H^{-1}(\{e\})$ is topologically mixing.*

Accordingly with this definition, we prove the following result.

Theorem 10 *There exists a residual set $\mathcal{R}$ in $C^2(M,\mathbb{R})$ such that, for any $H \in \mathcal{R}$, there is an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that, for every $e \in \mathcal{S}(H)$, the Hamiltonian level (H,e) is topologically mixing.*

The main tool to prove the previous result is a version for Hamiltonians of the Connecting Lemma for pseudo-orbits developed in [8] by Arnaud, Bonatti and Crovisier. To state it, we need the notions of *resonance relations* and of *pseudo-orbits*, which we postpone to Section 3.1.5 and Section 3.1.6.

Lemma 1 (Connecting Lemma for pseudo-orbits of Hamiltonians) *Let* (M, ω) *denote a compact, symplectic* $2d$*-manifold, for* $d \geq 2$. *Take* $H \in C^2(M, \mathbb{R})$ *and a regular energy* $e \in H(M)$*, such that the eigenvalues of any closed orbit of* H *do not satisfy non-trivial resonances. Then, for any* C^2*-neighborhood* $\mathcal{U}$ *of* H*, for any energy hypersurface* $\mathcal{E}_{H,e} \subset H^{-1}(\{e\})$ *and for any* $x, y \in \mathcal{E}_{H,e}$ *connected by an* ϵ*-pseudo-orbit, for* $\epsilon > 0$*, there exist* $\tilde{H} \in \mathcal{U}$ *and* $t > 0$ *such that* $e = \tilde{H}(x)$ *and* $X^t_{\tilde{H}}(x) = y$ *on the analytic continuation* $\mathcal{E}_{\tilde{H},e}$ *of* $\mathcal{E}_{H,e}$.

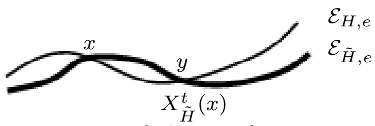

Figure 1.12: Representation of the analytic continuation of $\mathcal{E}_{H,e}$.

To prove these results, we have to resume the arguments used by Arnaud, Bonatti, Crovisier and Bessa in [8, 11, 24] and to adapt it to the Hamiltonian setting. The main change in the proofs is the need to restrict attention to the energy hypersurface, when analyzing the perturbations and their supports.

From Theorem 10, we can derive the following result concerning on the *homoclinic class* of a hyperbolic closed orbit γ of H, which is the closure of the set of transversal intersections between the stable and unstable manifolds of all points p in γ (see Section 3.1.4, for more details).

Corollary 4 *There is a residual set* $\mathcal{R}$ *in* $C^2(M, \mathbb{R})$ *such that, for any* $H \in \mathcal{R}$*, there is an open and dense set* $\mathcal{S}(H)$ *in* $H(M)$ *such that if* $e \in \mathcal{S}(H)$ *then any energy hypersurface of* $H^{-1}(\{e\})$ *is a homoclinic class.*

If any energy hypersurface of $H^{-1}(\{e\})$ is a homoclinic class, we say that $H^{-1}(\{e\})$ is a homoclinic class.

We end this chapter with an overview of the remaining chapters of this thesis. This thesis is organized in four additional chapters. In Chapter 2, we include the proofs of the results on conservative dynamics and in Chapter 3 we concern about the proofs of the results on Hamiltonian dynamics. In each chapter we also include extra definitions and useful auxiliary results. In the last chapters, we synthesize the main results of this thesis and we describe some ideas to improve and to develop this work.

CHAPTER
TWO

CONSERVATIVE DYNAMICS

This chapter begins with some extra definitions on conservative dynamics and it includes the statement of some auxiliary results. After, Section 2.2 brings together the complete proofs of the results on conservative dynamics, that is, of Theorem 1, Theorem 5, Theorem 7, Theorem 8, Theorem 9 and Corollary 1.

2.1 Definitions and auxiliary results

In this section, we state the definition of *Lyapunov exponents*, of the *Linear Poincaré flow* and of *heterodimensional cycles*. Afterwards, we state some perturbation results that will be used to complete the proofs, in Section 2.2.

2.1.1 Lyapunov exponents and classification of closed orbits

This section is about Lyapunov exponents for the conservative continuous-time case and their properties. Firstly, we remark that the Riemannian structure on M induces a norm $\|.\|$ on the fibers T_pM, $\forall\, p \in M$. From now on, we use the standard norm of a bounded linear map $L : TM \to TM$ given by

$$\|L\| = \sup_{\|u\|=1} \|L(u)\| \,.$$

Given $X \in \mathfrak{X}^1_\mu(M)$, Oseledets' theorem (see [59]) ensures that μ-almost every point $x \in M$ admits a splitting of the tangent bundle,

$$T_xM = E^1_x \oplus \cdots \oplus E^{k(x)}_x$$

and also real numbers $\lambda_1(x) > \cdots > \lambda_{k(x)}(x)$, for $1 \leq k(x) \leq d$, called *Lyapunov exponents*, such that $DX^t_x(E^i_x) = E^i_{X^t(x)}$ and

$$\lambda_i(x) = \lim_{t\to\pm\infty} \frac{1}{t} \log \|DX^t_x\,(v^i)\|,$$

for any $v^i \in E^i_x \setminus \{\vec{0}\}$ and $i \in \{1, ..., k(x)\}$. This splitting is called *Oseledets' splitting*. The full μ-measure set of *Oseledets' points* is denoted by $\mathcal{O}(X)$.

Remark 3 *As a consequence of Oseledets's theorem, we have that*

$$\sum_{i=1}^{k(x)} \lambda_i(x) \cdot \dim(E^i_x) = \lim_{t\to\pm\infty} \frac{1}{t} \log|\det DX^t_x|.$$

However, since the vector field X is divergence-free, we deduce that $|\det DX^t(x)| = 1$, for any $t \in \mathbb{R}$ and any $x \in M$. Therefore, we conclude that

$$\sum_{i=1}^{k(x)} \lambda_i(x) \cdot \dim(E^i_x) = 0, \;\; \forall\, x \in \mathcal{O}(X).$$

Note that if we do not take into account the multiplicities of the eigenvalues associated to the eigenspaces $E^1_x, \cdots, E^{k(x)}_x$, we have exactly $d = dim(M)$ Lyapunov exponents, $\lambda_1(x) \geq \cdots \geq \lambda_d(x)$.

Let $\gamma \subset M$ be a closed orbit of period π and fix $p \in \gamma$. The *characteristic multipliers* of γ are the eigenvalues of DX^π_p, which are independent of $p \in \gamma$. If σ is a characteristic multiplier of γ, then the associated Lyapunov exponent is $\lambda = \log(\sigma)/\pi$. A characteristic multiplier σ is said *simple* if its multiplicity is equal to 1.

Definition 2.1 *A closed orbit $\gamma \subset M$ is called*

- *hyperbolic, when all the characteristic multipliers have modulus different from 1;*
- *parabolic, when at least one of the characteristic multipliers is real and of modulus 1;*
- *completely elliptic, when all the characteristic multipliers are simple, non-real and of modulus 1;*
- *elliptic, when γ has at least two simple, non-real and of modulus 1 characteristic multipliers.*

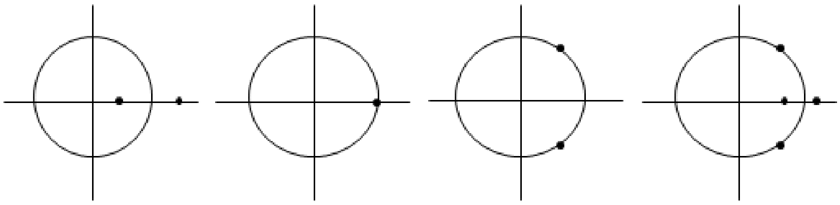

Figure 2.1: Representation of the spectrum of a hyperbolic, a parabolic, a completely elliptic and an elliptic closed orbit, respectivelly.

Notice that, given an *elliptic* or a *completely elliptic* closed orbit γ, if we do not assume the characteristic multipliers of γ to be simple then, under small perturbations, we are able to turn γ into a hyperbolic closed orbit. The same happens if we take a parabolic orbit.

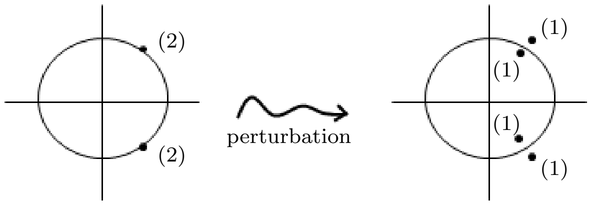

Figure 2.2: Transformation of a completely elliptic closed orbit, with no simple characteristic multipliers, into a hyperbolic closed orbit.

2.1.2 Linear Poincaré flow and hyperbolicity

In this section, we define the *linear Poincaré flow* and we state some results related with this flow. Let us start with some definitions.

Given X in $\mathfrak{X}^1(M)$ and a regular point x in M, let $N_x := X(x)^\perp \subset T_xM$ denote the $(dim(M) - 1)$–dimensional *normal bundle* of X at x and define $N_{x,r} := N_x \cap \{u \in T_xM : \|u\| < r\}$, for $r > 0$. Note that, in general, N_x is not DX^t_x-invariant.

Definition 2.2 *The flow $P^t_X(x) := \Pi_{X^t(x)} \circ DX^t_x$ is called linear Poincaré flow, where $\Pi_{X^t(x)} : T_{X^t(x)}M \to N_{X^t(x)}$ is the canonical orthogonal projection.*

Recently, Li, Gan and Wen generalized the notion of the linear Poincaré flow, in order to include singularities (see [48]).

Now, take in account the following result.

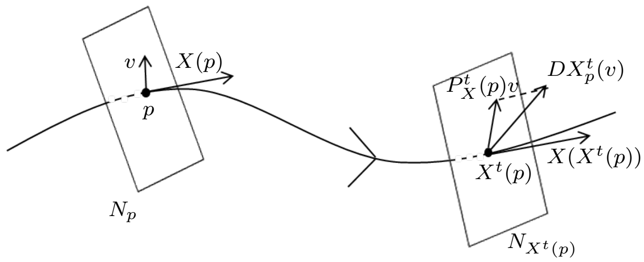

Figure 2.3: Representation of the linear Poincaré flow.

Lemma 2.1 *([56, Lemma 3.10]) Consider $X \in \mathfrak{X}^1(M)$ and $\Lambda \subset M$ a compact, X^t-invariant, regular set and assume that $E_\Lambda = E^1_\Lambda \oplus E^2_\Lambda$. If there exists $T > 0$ such that $\left\|DX^T_x|_{E^1_x}\right\| \leq 1/2$ and $\left\|DX^{-T}_{X^T(x)}|_{E^2_{X^T(x)}}\right\| \leq 1/2$, for every $x \in \Lambda$, then there are $c > 0$ and $0 < \kappa < 1$ such that $\left\|DX^t_x|_{E^1_x}\right\| < c\kappa^t$ and $\left\|DX^{-t}_{X^t(x)}|_{E^2_{X^t(x)}}\right\| < c\kappa^t$, for every $x \in \Lambda$ and $t > 0$.*

Taking into account the previous lemma, we state the following definition of *uniformly hyperbolic set* by using the linear Poincaré flow.

Definition 2.3 *Fix $X \in \mathfrak{X}^1(M)$. An X^t-invariant, compact and regular set $\Lambda \subset M$ is uniformly hyperbolic if N_Λ admits a P^t_X-invariant splitting $N^s_\Lambda \oplus N^u_\Lambda$ such that there is $\ell > 0$ satisfying*

$$\|P^\ell_X(x)|_{N^s_x}\| \leq \frac{1}{2} \text{ and } \|P^{-\ell}_X(X^\ell(x))|_{N^u_{X^\ell(x)}}\| \leq \frac{1}{2}, \text{ for any } x \in \Lambda.$$

Observe that the constant $\frac{1}{2}$ can be replaced by any constant $\theta \in (0,1)$. If θ is close to 1, we say that the hyperbolicity is weak.

Supported on an abstract invariant manifold theory result of Hirsch, Pugh and Shub (see [44, Lemma 2.18]), in [32] Doering proves that the definition of *uniformly hyperbolic compact set* by using the linear Poincaré flow (Definition 2.3) is equivalent to the usual definition of *uniformly hyperbolic set* of a flow (see Definition 1.2).

Lemma 2.2 *([32, Proposition 1.1]) Let Λ be a X^t-invariant, regular and compact set. Then Λ is uniformly hyperbolic for X^t if and only if Λ is uniformly hyperbolic for P^t_X.*

It is straightforward to see that the definition of Lyapunov exponent, stated in Section 2.1.1, can also be adapted in order to use P^t_X instead of DX^t. Hence, μ-almost every point $x \in M$ admits the Oseledets splitting

$$N_x = N_x^1 \oplus \cdots \oplus N_x^{k(x)},$$

for any $1 \leq k(x) \leq dim(M) - 1$, and the Lyapunov exponent

$$\lambda_i(x) = \lim_{t \to \pm\infty} \frac{1}{t} \log \|P_X^t(x)v^i\|,$$

for any $v^i \in N_x^i \setminus \{\vec{0}\}$ and $i \in \{1, ..., k(x)\}$.

A singularity p of a C^1-vector field X is *hyperbolic* if the eigenvalues of DX_p are not purely imaginary. In the divergence-free context, a hyperbolic critical point p must be of *saddle-type*. If p is a closed orbit then the dimension of the fibers N_p^s and N_p^u is between 1 and $dim(M) - 2$.

Now, we state the definition of *dominated splitting*, which is weaker that the definition of uniform hyperbolicity. For this, we use the linear Poincaré flow.

Definition 2.4 *Let $X \in \mathfrak{X}^1(M)$ and let $\Lambda \subset M$ be a compact, X^t-invariant and regular set. Assume that there exists a P_X^t-invariant splitting $N = N^1 \oplus \cdots \oplus N^k$ over Λ, for $1 \leq k \leq dim(M) - 1$, such that all the subbundles have constant dimension. This splitting is dominated if there exists $\ell > 0$ such that, for any $0 \leq i < j \leq k$,*

$$\|P_X^\ell(x)|_{N_x^i}\| \cdot \|P_X^{-\ell}(X^\ell(x))|_{N^j_{X^\ell(x)}}\| \leq \frac{1}{2}, \text{ for any } x \in \Lambda.$$

Note that a vector field with a dominated splitting structure is not necessarily uniformly hyperbolic.

Let us briefly state some useful properties of a dominated splitting over a set Λ (see [25] for more details):

- *Uniqueness*: the dominated splitting is unique, if the dimension of the subbundles is fixed.

- *Continuity*: any dominated splitting is continuous, that is, the subbundles N_x^1 and N_x^2 depend continuously on the point $x \in \Lambda$.

- *Transversality*: the angles between N^1 and N^2 are bounded away from zero on Λ.

- *Extension to the closure*: any ℓ-dominated splitting over a set Λ can be extended to an ℓ-dominated splitting over the closure of Λ.
- *Extension to a neighborhood*: the dominated splitting can be extended to the maximal flow-invariant set in a neighborhood of Λ.
- *Persistence*: any dominated splitting persists under C^1-perturbations.

Remark 4 *If we assume that there is not a dominated splitting on a flow-invariant, compact and regular set, it is possible to make a small C^1-perturbation on the vector field in order to get a new one with Lyapunov exponents arbitrarily close to zero, as it is shown by Bessa and Rocha in [17, Theorem 1].*

The next result corresponds to a dichotomy for C^1-divergence-free vector fields. It requires the existence of a closed orbit with arbitrarily large period. The proof of Theorem 2.1 for divergence-free vector fields follows the ideas stated in the proof of [19, Proposition 2.4].

Theorem 2.1 *Let $X \in \mathfrak{X}^1_\mu(M)$ and let $\mathcal{U}$ be a small C^1-neighborhood of X. Then, for any $\epsilon > 0$, there exist $l > 0$ and $\tau > 0$ such that, for any $Y \in \mathcal{U}$ and any $x \in Per^\tau(Y)$,*

- *either P^t_Y admits an l-dominated splitting over the Y^t-orbit of x, or else*
- *for any neighborhood U of x, there exists an ϵ-C^1-perturbation $\tilde{Y}$ of Y, coinciding with Y outside U and along the orbit of x, such that $P^{\pi(x)}_{\tilde{Y}}(x)$ has only eigenvalues equal to 1 or -1, where $\pi(x)$ stands for the period of x.*

The following result says that if the vector field has a linear hyperbolic singularity of saddle-type then the linear Poincaré flow cannot admit a dominated splitting over the set of regular points of M Note that a singularity p is *linear* if there exist smooth local coordinates around p such that X is linear and equal to DX_p in these coordinates (see [77, Definition 4.1]).

Proposition 2.1 *[77, Proposition 4.1] If $X \in \mathfrak{X}^1(M)$ has a linear hyperbolic singularity of saddle-type then P^t_X does not admit any dominated splitting over $M \backslash Sing(X)$.*

We remark that the proof of this proposition can be easily adapted to the conservative case. Hence, Proposition 2.1 remains valid for C^1-divergence-free vector fields.

We end this section with a lemma stating that a singularity can be turned into a linear one, by performing a small perturbation of the vector field.

Lemma 2.3 *[19, Lemma 3.3] Let p be a singularity of $X \in \mathfrak{X}^1_\mu(M)$. For any $\epsilon > 0$, there exists $Y \in \mathfrak{X}^\infty_\mu(M)$ such that Y is ϵ-C^1-close to X and p is a linear hyperbolic singularity of Y.*

2.1.3 Heterodimensional cycles

This section contains the definition of heterodimensional cycle, as well as some useful remarks.

Consider a C^1-vector field X and $p \in Crit(X)$. Denote by $\mathcal{O}_X(p)$ the X^t-orbit of p. We remark that if p is a singularity of X then we set $\mathcal{O}_X(p) = p$.

Definition 2.5 *Let X be a C^1-vector field and choose p in M. If $\mathcal{O}_X(p)$ is a hyperbolic set, its stable and unstable manifolds are defined as follows:*

$$W^s_X(\mathcal{O}_X(p)) = \{q \in M : \lim_{t\to+\infty} dist(X^t(q), \mathcal{O}_X(p)) = 0\} \quad \textit{and}$$

$$W^u_X(\mathcal{O}_X(p)) = \{q \in M : \lim_{t\to+\infty} dist(X^{-t}(q), \mathcal{O}_X(p)) = 0\}.$$

We observe that both $W^s_X(\mathcal{O}_X(p))$ and $W^u_X(\mathcal{O}_X(p))$ do not depend on $q \in \mathcal{O}_X(p)$. Therefore, we can write $W^s_X(\mathcal{O}_X(p)) = W^s_X(q)$ and $W^u_X(\mathcal{O}_X(p)) = W^u_X(q)$, for some $q \in \mathcal{O}_X(p)$. These manifolds are respectively tangent to the subspaces $E^s_q \oplus \mathbb{R}X(q)$ and $\mathbb{R}X(q) \oplus E^u_q$ of T_qM, for $q \in \mathcal{O}_X(p)$. Observe that

$$\dim(W^s_X(\mathcal{O}_X(p))) + \dim(W^u_X(\mathcal{O}_X(p))) = \dim(M) + i,$$

where $i = 0$ if $p \in Sing(X)$ and $i = 1$ if $p \in Per(X)$.

If $p \in Crit(X)$ is a hyperbolic saddle its *index* is defined as the dimension of the unstable bundle $W^u_X(p)$ and it is denoted by $ind(p)$.

Now, we state the notion of heterodimensional cycle for vector fields.

Definition 2.6 *Consider $X \in \mathfrak{X}^1(M)$ and let p, q be two distinct hyperbolic critical points of saddle-type such that $ind(p) < ind(q)$. A vector field X exhibits a heterodimensional cycle associated to p and q if the invariant manifolds of p and q intersect cyclically, that is $W^s_X(p) \pitchfork W^u_X(q) \neq \emptyset$ and $W^u_X(p) \cap W^s_X(q) \neq \emptyset$, where $\pitchfork$ denotes a transversal intersection. Let $\mathcal{HC}^1(M) \subset \mathfrak{X}^1(M)$ and $\mathcal{HC}^1_\mu(M) \subset \mathfrak{X}^1_\mu(M)$ denote the sets whose elements exhibit heterodimensional cycles.*

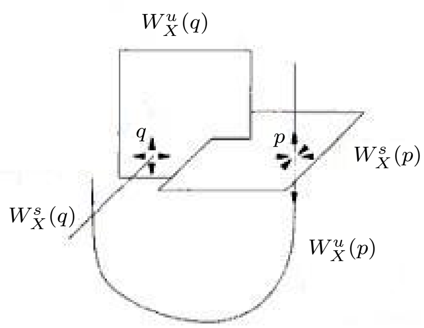

Figure 2.4: Representation of a heterodimensional cycle.

We observe that, for reasons of simplicity, the previous figure represents, in fact, a heterodimensional cycle for the discrete time case.

Remark 5 *The condition $ind(p) < ind(q)$, in Definition 2.6, ensures that the connection $W^s_X(p) \pitchfork W^u_X(q)$ is C^1-persistent and that the connection $W^u_X(p) \cap W^s_X(q)$ does not persist under C^1-generic perturbations.*

We observe that Definition 2.6 can be trivially extended to a finite number of hyperbolic saddles.

The next definition contains a classification of heterodimensional cycles.

Definition 2.7 *A heterodimensional cycle is called*

- *periodic, if it is composed just by closed orbits;*
- *singular, if it is composed just by singularities;*
- *mixed, if it contains at least one singularity and one closed orbit.*

Let us now state some appointments.

Remark 6 *We remark that*

- *if* $\dim(M) < 3$, M *does not support heterodimensional cycles because, in this case, we cannot find hyperbolic critical points of saddle-type with different indices;*

- *if* $\dim(M) = 3$, M *does not support periodic heterodimensional cycles. In this case, the stable and the unstable manifolds of any closed orbit are both* 2*-dimensional. However, it is possible to find singular and mixed heterodimensional cycles, where a link connecting two closed orbits is not allowed. Mixed heterodimensional cycles just appear in the case that the singularities have index* 1 *since, in this case, the index of any closed orbit is* 2*.*

We end this section with the definition of *far from heterodimensional cycles* vector fields.

Definition 2.8 *A vector field* $X \in \mathfrak{X}^1(M)$ *is far from heterodimensional cycles if there exists a* C^1*-neighborhood* $\mathcal{U}$ *of* X *in* $\mathfrak{X}^1(M)$ *such that any* $Y \in \mathcal{U}$ *does not exhibit heterodimensional cycles. If we assume* $X \in \mathfrak{X}^1_\mu(M)$*, the definition is analogous. Let* $\mathcal{FC}^1(M) \subset \mathfrak{X}^1(M)$ *and* $\mathcal{FC}^1_\mu(M) \subset \mathfrak{X}^1_\mu(M)$ *denote the sets whose elements are far from heterodimensional cycles.*

2.1.4 C^1-perturbation results

In this section, we state some useful perturbation lemmas for the conservative continuous-time case, namely the Zuppa Theorem, the C^1-*Closing Lemma*, the *Pasting Lemma* and the *Franks Lemma*.

The first perturbation result is due to Zuppa (see [81]) and it allows us to C^1-approximate any divergence-free vector field by a smooth one, keeping the divergence-free property.

Theorem 2.2 *The set of* C^∞*-divergence-free vector fields is* C^1*-dense in* $\mathfrak{X}^1_\mu(M)$*.*

The next result is a version of the C^1-*Closing Lemma* for divergence-free vector fields, firstly proved by Pugh and Robinson in [66]. More recently, this lemma was improved by Arnaud, who stated a simpler proof (see [7]). The C^1-Closing Lemma states that the orbit of a recurrent point, that is, a point which belongs to its ω-limit set, can be approximated by a long time closed orbit of a C^1-perturbation of the original vector field.

Lemma 2.4 *Consider $X \in \mathfrak{X}^1_\mu(M)$ and a X^t-recurrent point x. Given $\epsilon > 0, r > 0$ and $T > 0$, there exist an ϵ-C^1-neighborhood $\mathcal{U} \subset \mathfrak{X}^1_\mu(M)$ of X, a closed orbit p of $Y \in \mathcal{U}$, with arbitrarily large period π, a map $g : [0,T] \to [0,\pi]$, close to the identity, and $\tilde{T} > T$ such that*

- $d\big(X^t(x), Y^{g(t)}(p)\big) < \epsilon$, *for every* $0 \le t \le \tilde{T}$;
- $Y = X$ *on* $M \backslash B_r\big(X^{[0,\tilde{T}]}(x)\big)$.

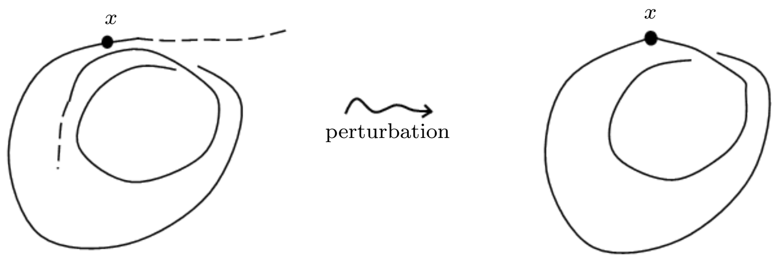

Figure 2.5: Perturbation given by the Closing Lemma.

A conservative version of Pugh and Robinson's *General Density Theorem* is stated in [66] and it is also proved by Arnaud in [7]. It asserts that, C^1-generically, the critical points of a vector field are dense in M.

Definition 2.9 *Let $\mathcal{PR}^1_\mu(M)$ denote the Pugh and Robinson residual set in $\mathfrak{X}^1_\mu(M)$.*

The Pasting Lemma (see [6]) allows us to realize C^1-local perturbations in the divergence-free setting. Its precise statement is as follows and when we say that Y is δ-C^1-close to X, we mean that $\|X - Y\|_{C^1} < \delta$.

Theorem 2.3 *Given $\epsilon > 0$, there exists $\delta > 0$ such that if $X \in \mathfrak{X}^1_\mu(M)$, $K \subset M$ is a compact set and $Y \in \mathfrak{X}^\infty_\mu(M)$ is δ-C^1-close to X in a small neighborhood $U \supset K$, then there exist $Z \in \mathfrak{X}^\infty_\mu(M)$ and open sets V and W, such that $K \subset V \subset U \subset W$, satisfying the properties:*

- $Z|_V = Y$;
- $Z|_{int(W^c)} = X$;
- Z *is* ϵ-C^1-*close to* X.

The last perturbation result is a version of Franks' Lemma for divergence-free vector fields (see [19]). Firstly, let us introduce the definition of *flowbox* and of *one-parameter linear family*.

Definition 2.10 *Take* $X \in \mathfrak{X}^1(M)$, $\tau > 0$ *and a regular point* $p \in M$ *such that* $X^t(p) \neq p$, *for any* $t \in [0, \tau]$, *and define the arc* $X^{[0,\tau]}(p) = \{X^t(p), t \in [0, \tau]\}$. *Fix* $r > 0$ *and* $\delta > 0$. *A flowbox is defined by*

$$\mathcal{T} := \mathcal{T}(p, \tau, r, \delta) = \bigcup_{t \in (-\delta, \tau+\delta)} X^t(B_r(p)),$$

where $B_r(p)$ *is chosen in a transversal section of* p.

The set $\mathcal{T}$ is an open neighborhood of $X^{[0,\tau]}(p)$. If $r > 0$ and $\delta > 0$, in the previous definition, are small enough, this neighborhood is foliated by regular orbits of the flow.

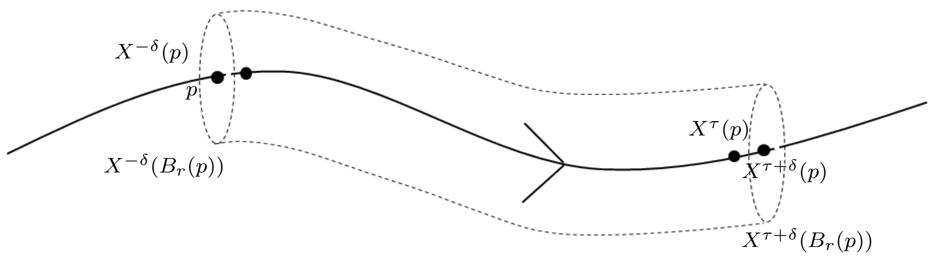

Figure 2.6: Representation of a flowbox.

Now, let $SL(d, \mathbb{R})$ denote the set of $d \times d$ matrices with determinant 1, with the group operation of ordinary matrix multiplication, where $d = \dim(M)$. Assume that p is as before and let $V, V' \subset T_pM$ be such that $\dim(V) = j$, $2 \leq j \leq d$ and $T_pM = V \oplus V'$.

Definition 2.11 *A one-parameter linear family* $\{A_t\}_{t\in\mathbb{R}}$ *associated to* $X^{[0,\tau]}(p)$ *and* V *is defined as follows:*

- $A_t : V \oplus V' \to V \oplus V'$ *is a linear map, for every* $t \in \mathbb{R}$, *such that*

$$A_t = \begin{cases} id & , t \leq 0 \\ A_\tau & , t \geq \tau \end{cases};$$

- *$A_t|_V \in SL(j, \mathbb{R})$, for $t \in [0, \tau]$;*
- *$A_t|_{V'} = id$, for $t \in [0, \tau]$, and $A_t(V) \subset V$;*
- *the family A_t is smooth on the parameter t.*

Note that $det(A_t) = 1$, for any $t \in \mathbb{R}$. Now, we state the Franks Lemma, which, under some conditions, allows us to realize a perturbation of the linear Poincaré flow of a given vector field as the linear Poincaré flow of a vector field which is C^1-close to the original one.

Theorem 2.4 *([19, Lemma 3.2]) Given $\epsilon > 0$ and a vector field $X \in \mathfrak{X}^4_\mu(M)$, there exists $\xi_0 = \xi_0(\epsilon, X)$ such that, for any $\tau \in [1, 2]$, for any $p \in Per^2(X)$, for any sufficient small flowbox $\mathcal{T}$ of $X^{[0,\tau]}(p)$ and for any one-parameter linear family $\{A_t\}_{t\in[0,\tau]}$ such that $\|A'_t A_t^{-1}\| < \xi_0$, for all $t \in [0, \tau]$, there exists $Y \in \mathfrak{X}^1_\mu(M)$ satisfying the following properties:*

- *Y is ϵ-C^1-close to X;*
- *$Y^t(p) = X^t(p)$, for any $t \in \mathbb{R}$;*
- *$P^\tau_Y(p) = P^\tau_X(p) \circ A_\tau$;*
- *$Y|_{\mathcal{T}^c} = X|_{\mathcal{T}^c}$.*

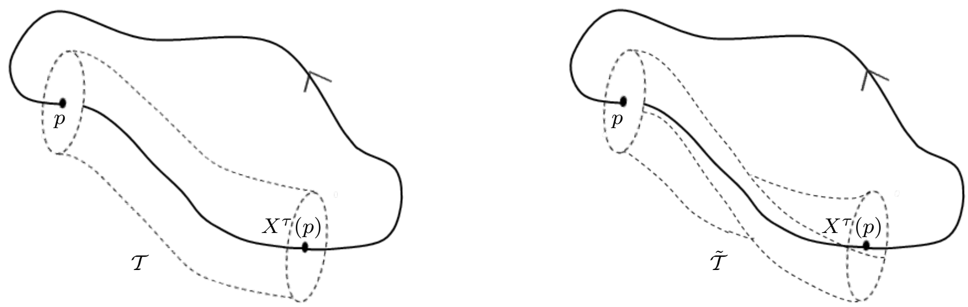

Figure 2.7: Representation of the action of the flow $P^\tau_Y(p)$.

Note that the constants 1 and 2, in the previous theorem, can be replaced by others. In fact, if the period π of the closed orbit is less than 2, we just have to redefine the length of the flowbox to be less that π in order to have enough time to perform the perturbation.

To complete this section, we refer a result due to Bessa. It asserts that, C^1-generically, a vector field is topologically mixing, and so transitive.

Theorem 2.5 *[11, Theorem 1.1] There exists a C^1-residual subset $\mathcal{R}$ of $\mathfrak{X}^1_\mu(M)$ such that if $X \in \mathcal{R}$ then X is a topologically mixing vector field.*

2.2 Proof of the conservative results

This section contains the proofs of Theorem 1, Theorem 5, Corollary 1, Theorem 7, Theorem 8 and Theorem 9.

2.2.1 Star property and uniform hyperbolicity

In this section we want to show that a C^1-divergence-free star vector field is uniformly hyperbolic.

Theorem 1 ([34, Theorem 1]) If $X \in \mathcal{G}^1_\mu(M^d)$ then $Sing(X) = \emptyset$ and $X \in \mathcal{A}^1_\mu(M^d)$, for $d \geq 4$.

The proof of this result is splitted in three main steps. First, we prove, in Lemma 2.5, that a C^1-divergence-free star vector field has no singularities. After that, in Lemma 2.6, we prove that the linear Poincaré flow admits a dominated splitting over the manifold. The last step consists on to reach uniform hyperbolicity from this domination, as shown in Lemma 2.8. For this, we prove an intermediate result (Lemma 2.7), which states that a divergence-free star vector field has uniform hyperbolicity on the period of closed orbits.

So, let us prove that a C^1-divergence-free star vector field does not have singularities.

Lemma 2.5 *If $X \in \mathcal{G}^1_\mu(M)$ then $Sing(X) = \emptyset$.*

Proof: Fix $X \in \mathcal{G}^1_\mu(M)$ and a C^1-neighborhood $\mathcal{U}$ of X in $\mathcal{G}^1_\mu(M)$, small enough such that Theorem 2.1 holds, that is, we have a dichotomy between a dominated splitting over a closed orbit, with arbitrarily large period, and the existence of a parabolic closed orbit for a vector field close to X.

Let $\mathcal{PR}^1_\mu(M)$ be the Pugh and Robinson residual set, described in Definition 2.9.

By contradiction, assume that there is $p \in Sing(X)$. Observe that, as $X \in \mathcal{G}^1_\mu(M)$, p is a hyperbolic saddle and so it persists to C^1-small perturbations of X. By Lemma 2.3, there is a smooth $Y \in \mathcal{U}$, C^1-close to X, such that p is a linear hyperbolic singularity of saddle-type of Y.

Now, choose a sequence of vector fields $Y_n \in \mathcal{U} \cap \mathcal{PR}^1_\mu(M)$, C^1-converging to Y. So, $M = \overline{Per(Y_n) \cup Sing(Y_n)}$, for any $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Since $Y_n \in \mathcal{G}^1_\mu(M)$, by the dichotomy in Theorem 2.1, there are positive constants ℓ and τ such that $P^t_{Y_n}$ admits an ℓ-dominated splitting over the closed orbits with period greater than τ. Since any closed orbit of Y_n is hyperbolic, note that $Per_\tau(Y_n)$ has a finite number of elements. Therefore, by the property of *extension to the closure* of a dominated splitting, $P^t_{Y_n}$ admits an ℓ-dominated splitting over the Y^t_n-invariant set $M \backslash Sing(Y_n)$. Taking a subsequence, if necessary, we can assume that the dimensions of the invariant bundles do not depend on n. So, the Y^t-invariant set

$$M \backslash Sing(Y) = \limsup_n \Big(M \backslash Sing(Y_n) \Big) = \bigcap_{N \in \mathbb{N}} \left(\overline{\bigcup_{n \geq N}^{\infty} M \backslash Sing(Y_n)} \right)$$

admits an ℓ-dominated splitting for P^t_Y.

However, since p is a linear hyperbolic singularity of saddle-type of Y, by Proposition 2.1, we conclude that P^t_Y does not admit a dominated splitting over $M \backslash Sing(Y)$. This is a contradiction. So, X has no singularities. $\square$

The next lemma states that the linear Poincaré flow associated to a divergence-free star vector field admits a dominated splitting over the manifold.

Lemma 2.6 *If $X \in \mathcal{G}^1_\mu(M)$ then P^t_X admits a dominated splitting $N = N^1 \oplus N^2$ over M.*

Proof: Consider $X \in \mathcal{G}^1_\mu(M)$ and a C^1-neighborhood $\mathcal{U}$ of X in $\mathcal{G}^1_\mu(M)$, small enough such that the dichotomy in Theorem 2.1 holds. Recall that, by the previous lemma, $Sing(X) = \emptyset$. Thus, P^t_X is well defined on M and there exists $\mathcal{V} \subset \mathcal{U}$, a C^1-neighborhood of X in $\mathcal{G}^1_\mu(M)$, whose elements do not have singularities.

By Theorem 2.1, since $X \in \mathcal{G}^1_\mu(M)$, there are positive constants ℓ and τ such that P^t_X admits an ℓ-dominated splitting over the X^t-orbit of any $p \in Per^\tau(X)$. Observe that, since any $x \in Per(X)$ is hyperbolic, we have the following P^t_X-invariant splitting

$N_x = N_x^s \oplus N_x^u$ such that any subbundle has constant dimension. In fact, if the dimension of the subbundles was not constant, as shown in Lemma 2.12 ahead, we would be able to construct a heterodimensional cycle, which is not allowed for star vector fields (see [37, Theorem 4.1]).

We claim that this splitting $N_x = N_x^s \oplus N_x^u$ is ℓ-dominated, for any $x \in Per^\tau(X)$. If this claim is not true, there is $q \in Per^\tau(X)$ such that the angle between N_q^s and N_q^u is arbitrarily close to 0 or such that q is weakly hyperbolic. In this situations, it is straightforward to see that, applying Zuppa's Theorem (Theorem 2.2) and Franks' Lemma (Theorem 2.4), we can C^1-perturb X in $\mathcal{V}$ in order to have Y such that q is a parabolic closed orbit of Y. But this is a contradiction, since $X \in \mathcal{G}_\mu^1(M)$. Therefore, any $p \in Per^\tau(X)$ admits the ℓ-dominated splitting $N_p = N_p^s \oplus N_p^u$.

Now, recall that a dominated splitting can be continuously extended to the closure of a set. Thus, the ℓ-dominated splitting over $Per^\tau(X)$ can be extended to $\overline{Per^\tau(X)}$. Observe that, given $X \in \mathcal{G}_\mu^1(M)$, the set $Per_\tau(X)$ has a finite number of elements. Hence, $\overline{Per^\tau(X)} = \overline{Per(X)}$. Finally, by [52, Lemma 3.1], since $X \in \mathcal{G}_\mu^1(M)$ has no singularities, we have that $\overline{Per(X)} = \Omega(X) = M$ and so, there is a dominated splitting $N = N^1 \oplus N^2$ over the manifold M. $\square$

See Appendix, for a different proof on the existence of a dominated splitting over M for a divergence-free star vector field.

Remark 7 *Observe that the previous lemma remains valid if we assume that X is an isolated point in the boundary of $\mathcal{A}_\mu^1(M)$. In fact, to prove Lemma 2.6, we use the fact that $X \in \mathcal{G}_\mu^1(M)$ to ensure the existence of dominated splitting over a closed orbit x, with arbitrarily large period π, for a vector field Y, C^1-close to X, given by Theorem 2.1. Therefore, if we start the proof by assuming that X is an isolated point in the boundary of $\mathcal{A}_\mu^1(M)$, we must obtain the same conclusion, because any C^1-perturbation $\tilde{Y}$ of Y must be Anosov, and so it cannot display a parabolic closed orbit.*

The following auxiliary result asserts that, for a divergence-free star vector field, any closed orbit is uniformly hyperbolic in the period. This is a crucial step to derive, from Lemma 2.6, uniform hyperbolicity on M.

Lemma 2.7 *Fix $X \in \mathcal{G}^1_\mu(M)$. There exist a C^1-neighborhood $\mathcal{U}$ of X in $\mathcal{G}^1_\mu(M)$ and a constant $\theta \in (0,1)$ such that, for any $Y \in \mathcal{U}$, if $p \in Per(Y)$ has period $\pi(p)$ and has the hyperbolic splitting $N_p = N^s_p \oplus N^u_p$ then:*

(a) $\|P_Y^{\pi(p)}(p)|_{N_p^s}\| < \theta^{\pi(p)}$ *and*

(b) $\|P_Y^{-\pi(p)}(p)|_{N_p^u}\| < \theta^{\pi(p)}$.

Proof: Fix $X \in \mathcal{G}^1_\mu(M)$ and a C^1-neighborhood $\mathcal{U}$ of X in $\mathcal{G}^1_\mu(M)$. So, for every $Y \in \mathcal{U}$, any $p \in Per(Y)$, with period $\pi(p)$, is a hyperbolic saddle. This means that $N_p = N^s_p \oplus N^u_p$ and that there is a constant $\theta_p \in (0,1)$ such that $\|P_Y^{\pi(p)}(p)|_{N_p^s}\| < \theta_p^{\pi(p)}$ and $\|P_Y^{-\pi(p)}(p)|_{N_p^u}\| < \theta_p^{\pi(p)}$. However, we want to prove that, in fact, we can choose θ_p not depending on p.

Let us prove (a). Suppose that, by contradiction, for any $\theta \in (0,1)$ there exist $Y \in \mathcal{U}$, C^1-arbitrarily close of X, and $p \in Per(Y)$, with period $\pi(p)$, hyperbolic by hypothesis, such that

$$\theta^{\pi(p)} \leq \|P_Y^{\pi(p)}(p)|_{N_p^s}\|.$$

In order to apply Theorem 2.4, we need Y to be a C^4-vector field. Therefore, applying Zuppa's theorem (Theorem 2.2), we start by C^1-approximate Y by a vector field $\tilde{Y} \in \mathcal{U} \cap \mathfrak{X}^4_\mu(M)$ such that $\tilde{\gamma}$ is a hyperbolic closed orbit of $\tilde{Y}$ and $\tilde{p} \in \tilde{\gamma}$ is the analytic continuation of p, so with period $\pi(\tilde{p})$ arbitrarily close to $\pi(p)$, and

$$\theta^{\pi(\tilde{p})} \leq \|P_{\tilde{Y}}^{\pi(\tilde{p})}(\tilde{p})|_{N_{\tilde{p}}^s}\|. \tag{2.1}$$

For simplicity, let us assume that the period $\pi(\tilde{p})$ is an integer. By the inequality in (2.1), we have that $\theta \leq \|P^1_{\tilde{Y}}(\tilde{q})|_{N^s_{\tilde{q}}}\|$, for some $\tilde{q} \in \mathcal{O}_{\tilde{Y}}(\tilde{p})$.

For $t \in [0, \pi(\tilde{p})]$, let A_t be a one-parameter family of linear maps, such that $\|A'_t A_t^{-1}\|$ is arbitrarily small, for any $t \in [0, \pi(\tilde{p})]$, and assume that $\|P^1_{\tilde{Y}}(\tilde{q})|_{N^s_{\tilde{q}}}\| = 1 - \rho$, where, by the relation (2.1), ρ is such that $0 < \rho < 1 - \theta$ and θ is chosen arbitrarily close to 1.

Now, define $A_t = id$, for $t \leq 0$, and, for $t \in [0, \pi(\tilde{p})]$, let A_t be a homothetic transformation of ratio of order $\dfrac{1}{1-\rho}$ and with entry $a_{1,n-1} = \delta\alpha(t)$, where $\alpha(t)$ is a smooth function such that $\alpha(t) = 1$, for $t \geq 1$, $\alpha(t) = 0$, for $t \leq 0$, $0 < \alpha'(t) < 1$, and

$\delta > 0$ is arbitrarily small. It is straightforward to see that $\|A'_t A_t^{-1}\| < \frac{\delta}{1-\rho}$ and that this norm can be taken arbitrarily small, by choosing $\delta > 0$ small enough.

Fix $\epsilon > 0$ and divide $\pi(\tilde{q})$ in $\pi(\tilde{q})$-one-time intervals. By Theorem 2.4, there are vector fields $Z_i \in \mathcal{G}^1_\mu(M)$, $\dfrac{\epsilon}{\pi(\tilde{q})}$-$C^1$-close to $\tilde{Y}$, such that $P^1_{Z_i}(\tilde{q}) = P^1_{\tilde{Y}}(\tilde{q}) \circ A_1$, for $i \in \{1, ..., \pi(\tilde{q})\}$. So, by the Pasting Lemma (Theorem 2.3), there exists $Z \in \mathcal{G}^1_\mu(M)$, ϵ-C^1-close to $\tilde{Y}$, such that $P_Z^{\pi(\tilde{q})}(\tilde{q})$ has an eigenvalue equal to 1 or -1. This is a contradiction because, since $Z \in \mathcal{G}^1_\mu(M)$, $\tilde{q}$ has to be a hyperbolic closed orbit of saddle-type.Then, (a) must hold. Item (b) is obtained using a similar argument. $\square$

Before to conclude the proof of Theorem 1, we state a remark.

Remark 8 *Fix a vector field X and a splitting $N = N^1 \oplus N^2$ over a compact manifold M. If $\liminf_{t\to+\infty} \|P^t_X(x)|_{N^1_x}\| = 0$ and $\liminf_{t\to+\infty} \|P^{-t}_X(x)|_{N^2_x}\| = 0$, for any $x \in M$, then M is hyperbolic (see [51] for more details).*

Now, by Lemma 2.7, we handle with the last step of the proof of Theorem 1.

Lemma 2.8 *If $X \in \mathcal{G}^1_\mu(M)$ is such that P^t_X admits a dominated splitting over M then M is uniformly hyperbolic.*

Proof: To prove this lemma, we adapt to the conservative setting a technique due to Mañé (see [51]). Let $X \in \mathcal{G}^1_\mu(M)$ be such that P^t_X admits the dominated splitting $N = N^1 \oplus N^2$ over M. By Lemma 2.5, $Sing(X) = \emptyset$. We want to prove that $P^t_X|_{N^1}$ is uniformly contracting on M and that $P^t_X|_{N^2}$ is uniformly expanding on M. Let us prove the first condition. By Remark 8, it suffices to prove that

$$\liminf_{t\to+\infty} \|P^t_X(x)|_{N^1_x}\| = 0, \, \forall\, x \in M.$$

By contradiction, suppose that there exists $x \in M$ satisfying

$$\liminf_{t\to+\infty} \|P^t_X(x)|_{N^1_x}\| > 0.$$

Therefore, we can choose a subsequence $\{t_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} t_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{1}{t_n} \log \|P^{t_n}_X(x)|_{N^1_x}\| \geq 0. \tag{2.2}$$

Let $C(M)$ denote the set of continuous functions on M and define $\varphi : C(M) \to \mathbb{R}$ by $\varphi(p) = \partial_h(\log \|P^h_X(p)|_{N^1_p}\|)_{h=0}$. By the Riez Theorem, there exists an X^t-invariant Borel probability measure μ such that

$$\begin{aligned}\int_M \varphi \, d\mu &= \lim_{t_n\to+\infty} \frac{1}{t_n}\int_0^{t_n} \varphi(X^s(x))\, ds\\ &= \lim_{t_n\to+\infty} \frac{1}{t_n}\int_0^{t_n} \partial_h(\log \|P^h_X(X^s(x))|_{N^1_{X^s(x)}}\|)_{h=0}\, ds\\ &= \lim_{t_n\to+\infty} \frac{1}{t_n} \log \|P^{t_n}_X(x)|_{N^1_x}\| \geq 0.\end{aligned}$$

Also, by the Birkhoff Ergodic Theorem,

$$\int_M \varphi \, d\mu = \int_M \lim_{t\to+\infty} \frac{1}{t}\int_0^t \varphi(X^s(x))\, ds d\mu(x) \geq 0.$$

Now, let $\Sigma(X)$ be the set of points $x \in M$ such that, for any C^1-neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^1_\mu(M)$ and any $\delta > 0$, there exist $Y \in \mathcal{U}$ and a Y-closed orbit $y \in M$ of period π such that $X = Y$ except on the δ-neighborhood of the Y-orbit of y, and that $dist(Y^t(y), X^t(x)) < \delta$, for $0 \leq t \leq \pi$. A conservative version of the Ergodic Closing Lemma, proved by Arnaud in [7], says that, given a X^t-invariant Borel probability measure μ, we have that $\mu(\Sigma(X)) = 1$. So, there is $x \in \Sigma(X)$ such that

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \varphi(X^s(x))\, ds = \lim_{t\to+\infty} \frac{1}{t} \log \|P^t_X(x)|_{N^1_x}\| \geq 0. \tag{2.3}$$

Let $\log\theta < \delta < 0$ be arbitrarily small, where $\theta \in (0,1)$ is fixed and given by Lemma 2.7. Thus, there is t_δ such that, for any $t \geq t_\delta$,

$$\frac{1}{t} \log \|P^t_X(x)|_{N^1_x}\| \geq \delta.$$

Now, since $x \in \Sigma(X)$, there exist $X_n \in \mathcal{U}$, C^1-converging to X, and $p_n \in Per(X_n)$ with period π_n. Notice that $\lim_{n\to+\infty} \pi_n = +\infty$, otherwise, by the relation in (2.3), we would have $x \in Per(X)$ with period π such that $P^\pi_X(x)|_{N^1_x}$ expands, which is a contradiction because $X \in \mathcal{G}^1_\mu(M)$. Thus, assuming that $\pi_n > t_\delta$, for every n, by continuity of the dominated splitting, we have that, for n big enough,

$$\|P^{\pi_n}_{X_n}(p_n)|_{N^1_{p_n}}\| \geq \exp(\delta\pi_n) > \theta^{\pi_n}.$$

But this contradicts (a) in Lemma 2.7, because $X_n \in \mathcal{U}$. So, $P^t_X|_{N^1}$ is uniformly contracting on M. Analogously, we prove that $P^t_X|_{N^2}$ is uniformly expanding on M, using (b) of Lemma 2.7. Hence, M is uniformly hyperbolic. $\square$

Combining the results in this section, we conclude that $\mathcal{G}^1_\mu(M) = \mathcal{A}^1_\mu(M)$.

2.2.2 Proof of the structural stability conjecture

In this section, we prove the C^1-structural stability conjecture for divergence-free vector fields.

Theorem 5 ([34, Theorem 2]) If $X \in \mathcal{SS}^1_\mu(M^d)$ then $X \in \mathcal{A}^1_\mu(M^d)$, for $d \geq 4$.

We start the proof of this result by showing that a C^1-structurally stable divergence-free vector field has no singularities.

Lemma 2.9 *If $X \in \mathcal{SS}^1_\mu(M)$ then $Sing(X) = \emptyset$.*

Proof: Let $X \in \mathfrak{X}^1_\mu(M)$ be a C^1-structurally stable vector field and let $\mathcal{V}$ be a small enough C^1-neighborhood of X in $\mathfrak{X}^1_\mu(M)$, such that any C^1-divergence-free vector field in $\mathcal{V}$ is topologically conjugated to X and the dichotomy in Theorem 2.1 holds for any $X \in \mathcal{V}$.

By contradiction, assume that X has a singularity p. By Lemma 2.3, we can find $Y \in \mathcal{V}$ such that p is a linear hyperbolic singularity of saddle-type for Y. Observe that the first part of the dichotomy stated in Theorem 2.1 cannot hold. In fact, as explained in Lemma 2.5, if there are positive constants τ and ℓ, such that P^t_Y admits an ℓ-dominated splitting over the Y^t-orbit of any $q \in Per^\tau(Y)$, then we conclude that $M \backslash Sing(Y)$ is ℓ-dominated. But this is not possible, by Proposition 2.1. Therefore, the second part of the dichotomy of Theorem 2.1 should work. However, since Y is topologically conjugated to $X \in \mathcal{SS}^1_\mu(M)$, we cannot find $Z \in \mathcal{V}$ such that $P^\pi_Z(x)$ has only eigenvalues equal to 1 or -1, for $x \in Per(Z)$ with arbitrarily large period π, because the existence of a parabolic closed orbit prevents the structural stability (see [70]). So, a C^1-structurally stable divergence-free vector field has no singularities. $\square$

Now, we are in conditions to go on with the proof of Theorem 5.

Fix $X \in \mathcal{SS}^1_\mu(M)$ and let $\mathcal{V}$ be a small enough C^1-neighborhood of X in $\mathfrak{X}^1_\mu(M)$, such that any C^1-divergence-free vector field in $\mathcal{V}$ is topologically conjugated to X. By contradiction, assume that X is not an Anosov divergence-free vector field. Therefore, by Theorem 1, $X \notin \mathcal{G}^1_\mu(M)$, meaning that, for any neighborhood $\mathcal{U}$ of X there exists

$Y \in \mathcal{U}$ such that Y has a non-hyperbolic critical point p. Choosing $\mathcal{U} = \mathcal{V}$, observe that, by Lemma 2.9, the vector field Y has, in fact, a non-hyperbolic closed orbit p. So, there exists a C^1-vector field $Y \in \mathcal{V}$, topologically conjugated to X, with a non-hyperbolic closed orbit p with period π. Hence $P_Y^\pi(p)$ has an eigenvalue with modulus 1. Now, by Zuppa's Theorem, there is a smooth $Z \in \mathcal{V}$, C^1-close to Y, such that $P_Z^\pi(p)$ also has an eigenvalue σ with modulus 1.

Remark 9 *In fact, $P_Z^\pi(p)$, in the proof, may not have an eigenvalue σ with modulus 1. In this case, observe that there exists $\mathcal{W} \subset \mathcal{U}$ and $\bar{Z} \in \mathcal{W}$, chosen C^1-arbitrarily close to Z and having an eigenvalue with modulus arbitrarily close to 1. So, by the Franks Lemma (Theorem 2.4), we can perform an ϵ-C^1-perturbation $\tilde{Z} \in \mathcal{W}$ of $\bar{Z}$, with $\epsilon > 0$ arbitrarily small, such that $P_{\tilde{Z}}^\pi(p)$ has an eigenvalue $\bar{\sigma}$ with $|\bar{\sigma}| = 1$.*

Accordingly with Moser's Theorem (see [58]), there is a smooth conservative change of coordinates $\varphi_p : U_p \to T_pM$ such that $\varphi_p(p) = \vec{0}$, where U_p is a small neighborhood of the closed orbit p. Let $f_Z : \varphi_p^{-1}(N_p) \to \Sigma$ be the Poincaré map associated to Z^t, where Σ denotes a Poincaré section through p, and $\mathcal{W}$ a C^1-neighborhood of f_Z. By the Franks Lemma (Theorem 2.4), taking $\mathcal{T}$ a small flowbox of $Z^{[0,t_0]}(p)$, with $0 < t_0 < \pi$, we have that there are $W \in \mathcal{V}$, $f_W \in \mathcal{W}$ and $\epsilon > 0$ such that:

- $W^t(p) = Z^t(p)$, for any $t \in \mathbb{R}$;
- $P_W^{t_0}(p) = P_Z^{t_0}(p)$;
- $W|_{\mathcal{T}^c} = Z|_{\mathcal{T}^c}$;
- for $\epsilon_0 > 0$ small,

$$f_W(x) = \begin{cases} \varphi_p^{-1} \circ P_Z^\pi(p) \circ \varphi_p(x) & , x \in B_{\epsilon_0}(p) \cap \varphi_p^{-1}(N_p) \\ f_Z(x) & , x \notin B_{4\epsilon_0}(p) \cap \varphi_p^{-1}(N_p). \end{cases}$$

Notice that $P_W^\pi(p)$ still has an eigenvalue σ satisfying $|\sigma| = 1$. First, assume that $\sigma = 1$. Let $K := \max_{0 \le i \le \pi} \|P_Z^i(p)\|$ and note that $K \ge 1$. Now, define

$$\mathcal{I}_v := \{sv : 0 \le s \le \epsilon_0/(2K)\}$$

and observe that $\|u\| \leq \dfrac{\epsilon_0}{2K} < \epsilon_0$, for any $u \in \mathcal{I}_v$, and $\|P_Z^i(p)u\| \leq K\epsilon_0/(2K) < \epsilon_0$, for any $0 \leq i \leq \pi$. Taking any $x \in \varphi_p^{-1}(\mathcal{I}_v)$, we have that $x = \varphi_p^{-1}(u)$, for some $u \in \mathcal{I}_v$. Thus,

$$\begin{aligned} f_W(x) = f_W(\varphi_p^{-1}(u)) &= \varphi_p^{-1} \circ P_Z^{\pi}(p) \circ \varphi_p(\varphi_p^{-1}(u)) \\ &= \varphi_p^{-1} \circ P_Z^{\pi}(p)u = \varphi_p^{-1}(u) = x. \end{aligned}$$

This means that any point in $\varphi_p^{-1}(\mathcal{I}_v)$ is a closed orbit of W with period less or equal than π. Recall that $W \in \mathcal{V}$ and so it is topologically conjugated to X. However, as shown by Robinson in [69], the set of C^1-Kupka-Smale divergence-free vector fields is a C^1-residual subset of $\mathfrak{X}_\mu^1(M)$. So, X must be topologically conjugated to a Kupka-Smale approximation, which has only a finite number of closed orbits with period less or equal than π, which is a contradiction.

Now, assume that $|\sigma| = 1$ but $\sigma \neq 1$. However, we point out that, by the Franks Lemma (Theorem 2.4), we can find $W \in \mathcal{V}$ such that $P_W^{\pi}(p)$ is a rational rotation. Then, there is $T \neq 0$ such that $P_W^{T+\pi}(p)$ has 1 as an eigenvalue. So, we can go on with the same argument. Hence, a C^1-structurally stable divergence-free vector field is Anosov, which concludes the proof of the structural stability conjecture for C^1-divergence-free vector fields. □

2.2.3 Boundary of $\mathcal{A}_\mu^1(M)$

In this section we prove that the boundary of the set of Anosov C^1-divergence-free vector fields has no isolated points.

Corollary 1 ([34, Corollary 1]) The boundary of the set $\mathcal{A}_\mu^1(M^d)$, for $d \geq 4$, has no isolated points.

Proof: By contradiction, assume that there exists an isolated vector field X on the boundary of the set $\mathcal{A}_\mu(M^d)$, for $d \geq 4$. In this case, we claim that $Sing(X) = \emptyset$. Let us assume that this claim is not true. If $p \in Sing(X)$ is hyperbolic, and so persistent to small C^1-perturbations of X, we can find a divergence-free vector field Y, arbitrarily close to X, such that $Sing(Y) \neq \emptyset$. But this is a contradiction because X is isolated on the boundary of $\mathcal{A}_\mu^1(M)$ and so Y has to be Anosov. If p is not hyperbolic, by

Lemma 2.3, we can transform p in a hyperbolic singularity of a vector field Z, C^1-close to X. Thus, as before, we reach a contradiction. So, $Sing(X) = \emptyset$ and, by Remark 7, P_X^t admits a dominated splitting over M. Therefore, we just have to follow the proof of Theorem 1, stated in Section 2.2.1, in order to conclude that $X \in \mathcal{A}_\mu^1(M)$, which is a contradiction. So, the boundary of the set $\mathcal{A}_\mu^1(M)$ cannot have isolated points. $\square$

2.2.4 Shadowing and uniform hyperbolicity

In this section, we prove that any divergence-free vector field in the C^1-interior of the set of divergence-free vector fields with the shadowing property is uniformly hyperbolic.

Theorem 7 ([33, Theorem 1]) If $X \in int(\mathcal{S}_\mu^1(M^d))$ then $X \in \mathcal{A}_\mu^1(M^d)$, for $d \geq 3$.

Let us start with the proof of a preliminary result. First, we prove that any divergence-free vector field in the C^1-interior of the set of divergence-free vector fields with the shadowing property has all the closed orbits hyperbolic (see Lemma 2.10). For this, we adapt the strategy described in [47] by Lee and Sakai. After this, we prove that a vector field with the described properties does not have singularities. Then, Theorem 7 follows immediately from Theorem 1.

Lemma 2.10 *If $X \in int(\mathcal{S}_\mu^1(M))$ then any closed orbit of X is hyperbolic.*

Proof: Take $X \in int(\mathcal{S}_\mu^1(M))$ and a C^1-neighborhood $\mathcal{U}$ of X in $\mathcal{S}_\mu^1(M)$. Let p be a point in a closed orbit γ of X with period π and U_p a small neighborhood of p on M. By contradiction, assume that there is an eigenvalue σ_0 of $P_X^\pi(p)$ satisfying $|\sigma_0| = 1$. Applying Zuppa's Theorem (Theorem 2.2), there is a smooth vector field $Y \in \mathcal{U}$ such that $Y^\pi(p) = p$. By Remark 9, recall that Y can be chosen such that $P_Y^\pi(p)$ has an eigenvalue σ with $|\sigma| = 1$.

Accordingly with Moser's Theorem (see [58]), there is a smooth conservative change of coordinates $\varphi_p : U_p \to T_pM$ such that $\varphi_p(p) = \vec{0}$. Recall that $f_Y : \varphi_p^{-1}(N_p) \to \Sigma$ denotes the Poincaré map associated to Y^t, where Σ is a Poincaré section through p. Let $\mathcal{V}$ be a C^1-neighborhood of f_Y. By the Franks Lemma (Theorem 2.4), taking $\mathcal{T}$ a small flowbox of $Y^{[0,t_0]}(p)$, with $0 < t_0 < \pi$, there are $Z \in \mathcal{U}$, $f_Z \in \mathcal{V}$ and $\epsilon > 0$ such that:

- $Z^t(p) = Y^t(p)$, for any $t \in \mathbb{R}$;
- $P_Z^{t_0}(p) = P_Y^{t_0}(p)$;
- $Z|_{\mathcal{T}^c} = Y|_{\mathcal{T}^c}$;
- $$f_Z(x) = \begin{cases} \varphi_p^{-1} \circ P_Y^{\pi}(p) \circ \varphi_p(x) & , x \in B_{\epsilon_0}(p) \cap \varphi_p^{-1}(N_p) \\ f_Y(x) & , x \notin B_{4\epsilon_0}(p) \cap \varphi_p^{-1}(N_p), \end{cases}$$
  where $\epsilon_0 > 0$ is small.

Notice that $P_Z^{\pi}(p)$ still has an eigenvalue σ with modulus 1. Firstly, assume that $\sigma = 1$, fix the associated non-zero eigenvector v such that $\|v\| = \epsilon_0/2$ and define

$$\mathcal{I}_v := \{sv : 0 \leq s \leq 1\}.$$

Since $Z \in \mathcal{S}_\mu^1(M)$, for any $\epsilon > 0$, there is $\delta > 0$ such that any (δ, T)-pseudo-orbit is ϵ-shadowed by some orbit y of Z^t, for $T > 0$. Fix $0 < \epsilon < \dfrac{\epsilon_0}{4}$. The idea now is to construct a (δ, T)-pseudo-orbit of Z^t, adapting the strategy followed by Lee and Sakai in [47, Proposition A]. Let us present the highlights of that proof.

Let $x_0 = p$ and $t_0 = 0$. Since p is a parabolic closed orbit, we construct a finite sequence $\{(x_i, t_i)\}_{i=0}^{I}$, where $I \in \mathbb{N}$, $t_i > 0$ and $x_i \in \varphi_p^{-1}(\mathcal{I}_v)$, for $1 \leq i \leq I$, such that:

- $x_I = \varphi_p^{-1}(v)$;
- $dist(Z^t(f_Z(x_i)), Z^t(x_{i+1})) < \delta$, for $|t| \leq T$ and $0 \leq i \leq I - 1$;
- $Z^{t_i}(x_i) = f_Z(x_i)$, for $1 \leq i \leq I$.

So, taking $S_n := \sum_{i=0}^{n} t_i$, for $0 \leq n \leq I$, the map $\psi : \mathbb{R} \to M$ defined by

$$\psi(t) = \begin{cases} Z^t(x_0) & , t < 0 \\ Z^{t-S_n}(x_{n+1}) & , S_n \leq t < S_{n+1},\ 0 \leq n \leq I - 2 \\ Z^{t-S_{I-1}}(x_I) & , t \geq S_{I-1}, \end{cases}$$

is a (δ, T)-pseudo-orbit of Z^t. So, since $Z \in \mathcal{U}$, there is a reparametrization $\alpha \in Rep(\epsilon)$ and a point $y \in B_\epsilon(p) \cap \varphi_p^{-1}(N_{p,\epsilon})$ that ϵ-shadows ψ. So, $dist(Z^{\alpha(t)}(y), \psi(t)) < \epsilon$, for any $t \in \mathbb{R}$. Note that, since $\sigma = 1$,

$$dist(x_0, x_I) = dist\big(p, \varphi_p^{-1}(v)\big) = dist\big(p, f_Z(\varphi_p^{-1}(v))\big) = \|v\| = \frac{\epsilon_0}{2} > 2\epsilon.$$

However, since Z has the shadowing property,

$$dist(x_0, x_I) \leq dist\big(x_0, Z^{\alpha(S_{I-1})}(y)\big) + dist\big(Z^{\alpha(S_{I-1})}(y), \psi(S_{I-1})\big) < 2\epsilon,$$

which is a contradiction.

Now, if $|\sigma| = 1$ but $\sigma \neq 1$, we point out that, by Theorem 2.4, we can find $W \in \mathcal{U}$ such that $P^{\pi}_{W}(p)$ is a rational rotation. Then, there is $T \neq 0$ such that $P^{T+\pi}_{W}(p)$ has 1 as an eigenvalue. Therefore, reproducing the previous argument, we conclude that any closed orbit of $X \in int(\mathcal{S}^1_\mu(M))$ is hyperbolic. $\square$

Now, by Theorem 1, if we show that any divergence-free vector field in the C^1-interior of the set $\mathcal{S}^1_\mu(M)$ has no singularities, we conclude the proof of Theorem 7. Let us prove it.

Take $X \in int(\mathcal{S}^1_\mu(M))$ and let $\mathcal{U}$ be a C^1-neighborhood of X in $\mathcal{S}^1_\mu(M)$, small enough such that the dichotomy of Theorem 2.1 holds.

By contradiction, assume that $Sing(X) \neq \emptyset$ and fix $p \in Sing(X)$. By Lemma 2.3, there is $Y \in \mathcal{U}$ such that $p \in Sing(Y)$ is linear hyperbolic, and so of saddle-type. Hence, by Proposition 2.1, P^t_Y does not admit any dominated splitting over $M \backslash Sing(Y)$. However, since any closed orbit of Y is hyperbolic (Lemma 2.10), it is straightforward to see that, reproducing the techniques used in the proof of Lemma 2.5, P^t_Y admits a dominated splitting over $M \backslash Sing(Y)$. Therefore, $Sing(X) = \emptyset$. $\square$

2.2.5 Expansiveness and uniform hyperbolicity

In this section we show that a divergence-free vector field in the C^1-interior of the set of expansive divergence-free vector fields is uniformly hyperbolic.

Theorem 8 ([33, Theorem 1]) If $X \in int(\mathcal{E}^1_\mu(M^d))$ then $X \in \mathcal{A}^1_\mu(M^d)$, for $d \geq 3$.

The proof of Theorem 8 follows a similar strategy to that one described in the previous section. Thus, let us start with the proof of the following result, based on the ideas of Moriyasu, Sakai and Sun, in [57].

Lemma 2.11 *If $X \in int(\mathcal{E}^1_\mu(M))$ then any closed orbit of X is hyperbolic.*

Proof: Consider $X \in int(\mathcal{E}^1_\mu(M))$ and a C^1-neighborhood $\mathcal{U}$ of X in $\mathcal{E}^1_\mu(M)$. Let p be a point in a closed orbit γ of X with period π and U_p a small neighborhood of p on M.

By contradiction, assume that there is an eigenvalue σ_0 of $P^{\pi}_X(p)$ such that $|\sigma_0| = 1$. Applying Zuppa's Theorem (Theorem 2.2), there is $Y \in \mathcal{U}$ such that $Y \in \mathfrak{X}^{\infty}_{\mu}(M)$, $Y^{\pi}(p) = p$ and $P^{\pi}_Y(p)$ has an eigenvalue σ such that $|\sigma| = 1$, as explained in Remark 9.

Let φ and f_Y be as in the proof of Lemma 2.10 and fix a C^1-neighborhood $\mathcal{V}$ of f_Y. By the Franks Lemma (Theorem 2.4), taking $\mathcal{T}$ a small flowbox of $Y^{[0,t_0]}(p)$, with $0 < t_0 < \pi$, there are $Z \in \mathcal{U}$ and $f_Z \in \mathcal{V}$ such that:

- $Z^t(p) = Y^t(p)$, for any $t \in \mathbb{R}$;
- $P^{t_0}_Z(p) = P^{t_0}_Y(p)$;
- $Z|_{\mathcal{T}^c} = Y|_{\mathcal{T}^c}$;
- $$f_Z(x) = \begin{cases} \varphi_p^{-1} \circ P^{\pi}_Y(p) \circ \varphi_p(x) & , x \in B_{\epsilon/4}(p) \cap \varphi_p^{-1}(N_p) \\ f_Y(x) & , x \notin B_{\epsilon}(p) \cap \varphi_p^{-1}(N_p). \end{cases}$$

Observe that $P^{\pi}_Z(p)$ still has an eigenvalue σ with modulus 1.

Since $Z \in \mathcal{E}^1_{\mu}(M)$, for a sufficiently small $\epsilon > 0$, there is $0 < \delta < \epsilon$ such that, if $x, y \in M$ satisfy $dist(Z^t(x), Z^{\alpha(t)}(y)) \leq \delta$, for any $t \in \mathbb{R}$ and for some continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $y = Z^s(x)$, where $|s| \leq \epsilon$. So, take $0 < \delta' < \delta$ such that if $x, y \in M$ satisfy $dist(x, y) < \delta'$ then $dist(Z^t(x), Z^t(y)) < \delta$, for $0 \leq t \leq \pi$.

As shown in the proof of Lemma 2.10, it is enough to assume that the eigenvalue σ is equal to 1. Fix a non-zero eigenvector v associated to σ such that $\|v\| < \delta'$. Now, choose $\varphi_p^{-1}(v) \in \varphi_p^{-1}(N_p)\backslash\{p\}$ and observe that

$$f_Z(\varphi_p^{-1}(v)) = \varphi_p^{-1} \circ P^{\pi}_Y(p) \circ \varphi_p(\varphi_p^{-1}(v)) = \varphi_p^{-1} \circ P^{\pi}_Y(p)(v) = \varphi_p^{-1}(v).$$

Thus, $dist(p, \varphi_p^{-1}(v)) = dist(p, f_Z(\varphi_p^{-1}(v))) = \|v\| < \delta'$ and, by the choice of δ', we have that $dist(Z^t(p), Z^t(\varphi_p^{-1}(v))) < \delta$, for any $0 \leq t \leq \pi$. Then, there is a continuous function $\alpha : \mathbb{R} \to \mathbb{R}$, with $\alpha(0) = 0$, such that $dist(Z^t(p), Z^{\alpha(t)}(\varphi_p^{-1}(v))) < \delta$, for every $t \in \mathbb{R}$. Since $Z \in \mathcal{E}^1_{\mu}(M)$, we have that $\varphi_p^{-1}(v) = Z^s(p)$, for $|s| \leq \epsilon$. This is a contradiction, because $\varphi_p^{-1}(v) \in \varphi_p^{-1}(N_p)\backslash\{p\}$. Hence, any closed orbit of X in $int(\mathcal{E}^1_{\mu}(M))$ is hyperbolic. $\square$

Now, we remark that, in [28, Lemma 1], Bowen and Walters prove that if $p \in M$ is a singularity of an expansive vector field then there is $\epsilon > 0$ such that $B_{\epsilon}(p) = \{p\}$.

Therefore, since M is a connected manifold, M must be regular. So, in particular, if $X \in int(\mathcal{E}^1_\mu(M))$ then $Sing(X) = \emptyset$. Anyway, as explained before, we can also adapt the proof of Lemma 2.5 in order to prove that M is regular. Hence, by Theorem 1, $int(\mathcal{E}^1_\mu(M)) \subset \mathcal{A}^1_\mu(M)$.

2.2.6 Heterodimensional cycles and uniform hyperbolicity

In this section, we show that divergence-free vector fields with a heterodimensional cycle are C^1-dense in the complement of the C^1-closure of Anosov divergence-free vector fields.

Theorem 9 ([34, Theorem 3]) If $X \in \mathfrak{X}^1_\mu(M^d)$, for $d \geq 4$, then X can be C^1-approximated by an Anosov divergence-free vector field, or else by a divergence-free vector field exhibiting a heterodimensional cycle.

In order to prove this result, we start by showing two auxiliary results. The first one states that, C^1-generically, a far from heterodimensional cycles divergence-free vector field has all the critical points hyperbolic and with constant index. This happens because, if we allow the existence of a C^1-generic divergence-free vector field having two critical points with different indices, we can perturb it in order to construct a heterodimensional cycle. Recall that $\mathcal{FC}^1_\mu(M)$ denotes the set of far from heterodimensional cycles divergence-free vector fields and that $\mathcal{KS}^1_\mu(M)$ is the Kupka-Smale residual set in $\mathfrak{X}^1_\mu(M)$.

Lemma 2.12 *There exists a residual set $\mathcal{S} \subset \mathcal{FC}^1_\mu(M)$ such that, for any $X \in \mathcal{S}$, all the critical points of X are hyperbolic and their index is constant.*

Proof: Let $\mathcal{S} := \mathcal{FC}^1_\mu(M) \cap \mathcal{KS}^1_\mu(M)$, which is a C^1-residual subset of $\mathcal{FC}^1_\mu(M)$. Observe that, by definition of $\mathcal{S}$, any critical point of $X \in \mathcal{S}$ is hyperbolic. Consider $X \in \mathcal{S}$ with two hyperbolic critical points Δ_X and Γ_X with different indices, say $ind(\Delta_X) < ind(\Gamma_X)$. Notice that Δ_X and Γ_X can be closed orbits or singularities. Fix $p_X \in \Delta_X$ and $q_X \in \Gamma_X$. Let $\mathcal{U}$ be an arbitrarily small C^1-neighborhood of X in $\mathfrak{X}^1_\mu(M)$, such that the analytic continuation of p_X and q_X, say p_Y and q_Y, is well defined for any $Y \in \mathcal{U}$.

By Theorem 2.5, there exists a topologically mixing $Y \in \mathcal{U} \cap \mathcal{S}$. Hence, since M is compact, Y has a dense orbit on M. So, fixing $p \in W^s_Y(p_Y)$ and $q \in W^u_Y(q_Y)$, Y has an orbit which passes arbitrarily close to p and q. Therefore, applying the conservative version of the Connecting Lemma for flows (see [79]), there exists $\bar{Y} \in \mathcal{U} \cap \mathcal{S}$ such that $W^s_{\bar{Y}}(p_{\bar{Y}})$ and $W^u_{\bar{Y}}(q_{\bar{Y}})$ intersect transversely. Repeating the previous argument, we obtain $\tilde{Y} \in \mathcal{U} \cap \mathcal{S}$, C^1-arbitrarily close to $\bar{Y}$, such that $W^u_{\tilde{Y}}(p_{\tilde{Y}}) \cap W^s_{\tilde{Y}}(q_{\tilde{Y}}) \neq \emptyset$, but also $W^s_{\tilde{Y}}(p_{\tilde{Y}}) \cap W^u_{\tilde{Y}}(q_{\tilde{Y}}) \neq \emptyset$. This happens because the first connection is C^1-robust and so it persists to small C^1-perturbations. Thus, $\tilde{Y} \in \mathcal{S}$ exhibits a heterodimensional cycle, which can be a periodic, a singular or a mixed heterodimensional cycle. But this is a contradiction, because $X \in \mathcal{FC}^1_\mu(M)$. Then, any critical element of X in $\mathcal{S}$ is hyperbolic and has constant index. $\square$

The next lemma allows us to prove that a far from Anosov C^1-divergence-free vector field can be C^1-approximated by a divergence-free vector field exhibiting a heterodimensional cycle.

Lemma 2.13 *If $X \in \mathfrak{X}^1_\mu(M) \backslash \overline{\mathcal{A}^1_\mu(M)}$ then X can be C^1-approximated by a divergence-free vector field with a heterodimensional cycle.*

Proof: Assume that $X \in \mathfrak{X}^1_\mu(M) \backslash \overline{\mathcal{A}^1_\mu(M)}$. So, by Theorem 1, X belongs to $\mathfrak{X}^1_\mu(M) \backslash \overline{\mathcal{G}^1_\mu(M)}$. So, for any $Y \in \left(\mathfrak{X}^1_\mu(M) \backslash \overline{\mathcal{G}^1_\mu(M)}\right) \cap \mathcal{KS}^1_\mu(M) \cap \mathcal{PR}^1_\mu(M)$, C^1-arbitrarily close to X, there exists a hyperbolic closed orbit p_Y of Y, with period π_Y and index u. Let $\mathcal{W}$ be a small C^1-neighborhood of Y such that the analytic continuation of p_Y, say p_Z, is well defined for any $Z \in \mathcal{W}$.

As Y belongs to the open set $\mathfrak{X}^1_\mu(M) \backslash \overline{\mathcal{G}^1_\mu(M)}$, for any C^1-neighborhood $\mathcal{V}$ of Y in $\mathfrak{X}^1_\mu(M) \backslash \overline{\mathcal{G}^1_\mu(M)}$, there is a vector field $Z \in \mathcal{W} \cap \mathcal{V}$, C^1-arbitrarily close to Y, such that Z has a hyperbolic closed orbit p_Z, with period π_Z close to π_Y and index u, corresponding to the analytic continuation of p_Y. However, since $Z \in \mathcal{V}$, it has a non-hyperbolic critical point q_Z, which can be a singularity or a closed orbit. Let us analyze both cases separately.

If q_Z is a non-hyperbolic singularity of Z, by a C^1-small perturbation of Z, it can become in a hyperbolic singularity with index $v \neq u$. Observe that this perturbation can produce different non-hyperbolic critical points but it does not matter, since we already have two hyperbolic critical points with different indices. So, as shown in Lemma 2.12, we are able to construct a heterodimensional cycle.

Now, assume that q_Z is a non-hyperbolic closed orbit of Z. In this case, we start by applying Zuppa's Theorem (Theorem 2.2) to increase the differentiability of the vector field Z from C^1 to C^4, in order to apply Theorem 2.4, which ensures the existence of a vector field $W \in \mathfrak{X}^4_\mu(M) \cap \mathcal{W}$, C^1-close to Y, such that p_W and q_W are now hyperbolic closed orbits with different indices. Again, by Lemma 2.12, we can C^1-approximate W by a vector field exhibiting a heterodimensional cycle. $\square$

By the previous two lemmas, the conclusion of the proof of Theorem 9 becomes really simple. It is enough to show that $X \in \mathcal{FC}^1_\mu(M)$ can be C^1-approximated by an Anosov divergence-free vector field. So, take $X \in \mathcal{FC}^1_\mu(M)$. By Lemma 2.13, $\mathcal{FC}^1_\mu(M) \subset \overline{\mathcal{G}^1_\mu(M)}$. Since $\mathcal{FC}^1_\mu(M)$ is open in $\mathfrak{X}^1_\mu(M)$, X can be C^1-approximated by a divergence-free vector field $Y \in \mathcal{FC}^1_\mu(M) \cap \mathcal{G}^1_\mu(M)$. Finally, Theorem 1 ensures that Y is Anosov, which concludes the proof. $\square$

CHAPTER

THREE

HAMILTONIAN DYNAMICS

This chapter contains extra definitions and some auxiliary results on Hamiltonian dynamics. Afterwards, we prove Lemma 1, Theorem 2, Theorem 3, Theorem 4, Theorem 6, Theorem 10, Corollary 2 and Corollary 4.

3.1 Definitions and auxiliary results

This section starts with the presentation of some more definitions on *Hamiltonian dynamics*. After, we define the *transversal Linear Poincaré flow* and we include some notes on *topological dimension*. We also describe, for Hamiltonians, *homoclinic classes*, *resonance relations*, *pseudo-orbits*, *perturbation flowboxes*, *covering families* and *avoidable closed orbits*. We end this section with the statement of some perturbation results, that will be used in Section 3.3.

3.1.1 Some notes on Hamiltonian dynamics

Recall that (M, ω) denotes a symplectic manifold, where M is an even-dimensional manifold endowed with a symplectic form ω. Recall that a symplectic form is a closed, bilinear, skew-symmetric and non-degenerate 2-form on the tangent bundle TM. These properties, on the symplectic form, play an important role in the characterization of the Hamiltonian dynamics. The non-degeneracy of the form ω guarantees that a Hamiltonian vector field is well-defined, while the skew-symmetry of ω leads to conservative properties for the Hamiltonian vector field. Once more, since ω is non-degenerate, given H in

$C^2(M,\mathbb{R})$ and $p \in M$, we know that $d_pH = 0$ is equivalent to $X_H(p) = 0$, where d_pH stands for the gradient of H in $p \in M$. Therefore, the extreme values of a Hamiltonian H are exactly the singularities of the associated Hamiltonian vector field X_H. Let $Per(H)$ denote the set of closed orbits of X_H and $Sing(H)$ denote the set of singularities of X_H.

We say that $\tilde{H}$ is $\epsilon - C^2$-close to H, for $\epsilon > 0$ fixed, if $\|H - \tilde{H}\|_{C^2} < \epsilon$, where $\|H - \tilde{H}\|_{C^2}$ denotes the C^2-distance between H and $\tilde{H}$.

Given a Hamiltonian level (H,e), let $\Omega(H|_{\mathcal{E}_{H,e}})$ be the set of non-wandering points of H on the energy hypersurface $\mathcal{E}_{H,e}$, that is, the points $x \in \mathcal{E}_{H,e}$ such that, for every neighborhood U of x in $\mathcal{E}_{H,e}$, there is $T > 0$ such that $X_H^T(U) \cap U \neq \emptyset$.

Fix a Hamiltonian level (H,e). As mentioned in Chapter 1, we want $H^{-1}(\{e\})$ to decompose into a finite number of connected components, say $H^{-1}(\{e\}) = \sqcup_{i=1}^{I_e} \mathcal{E}_{H,e,i}$, for $I_e \in \mathbb{N}$. Let us look at the following example.

Example 1: Write $H : \mathbb{R}^2 \to \mathbb{R}$ such that

$$H(x,y) = \begin{cases} x^7 \sin\left(\dfrac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0. \end{cases}$$

It is immediate to see that, for $e = 0$, $H^{-1}(\{e\})$ corresponds to an infinite number of connected components. This construction can be made local. A direct consequence of the Implicit Function Theorem ensures that the absence of singularities is enough to ensure a finite decomposition of $H^{-1}(\{e\})$.

By *Liouville's Theorem*, the symplectic manifold (M,ω) is also a volume manifold (see, for example, [2]). This means that the volume form $\omega^2 = \omega \wedge \omega$ induces a measure μ on M, which is the Lebesgue measure associated to ω^2. Notice that the measure μ on M is preserved by the Hamiltonian flow. So, given a regular Hamiltonian level (H,e), we induce a volume form $\omega_{\mathcal{E}_{H,e}}$ on each energy hypersurface $\mathcal{E}_{H,e} \subset H^{-1}(\{e\})$:

$$\begin{aligned} \omega_{\mathcal{E}_{H,e}} : T_p\mathcal{E}_{H,e} \times T_p\mathcal{E}_{H,e} \times T_p\mathcal{E}_{H,e} &\longrightarrow \mathbb{R} \\ (u,v,w) &\longmapsto \omega^2(d_pH, u, v, w), \;\; \forall\, p \in \mathcal{E}_{H,e}. \end{aligned}$$

The volume form $\omega_{\mathcal{E}_{H,e}}$ is X_H^t-invariant. Hence, it induces an invariant volume measure $\mu_{\mathcal{E}_{H,e}}$ on $\mathcal{E}_{H,e}$ that is finite, since any energy hypersurface is compact. Observe that, under these conditions, we have that $\mu_{\mathcal{E}_{H,e}}$-a.e. $x \in \mathcal{E}_{H,e}$ is recurrent, by the *Poincaré Recurrence Theorem*.

Definition 3.1 *We say that the measure $\mu_{\mathcal{E}_{H,e}}$ is ergodic if, for any X_H^t-invariant subset S of $\mathcal{E}_{H,e}$, we have that $\mu_{\mathcal{E}_{H,e}}(S) = 0$ or $\mu_{\mathcal{E}_{H,e}}(S) = 1$.*

Now we state the definition of *transitive Hamiltonian level*, which is weaker than the definition of topologically mixing Hamiltonian level (Definition 1.17).

Definition 3.2 *A Hamiltonian vector field X_H, restricted to a energy hypersurface $\mathcal{E}_{H,e}$, is transitive if, for any open and non-empty subsets U and V of $\mathcal{E}_{H,e}$, there is $\tau \in \mathbb{R}$ such that $X_H^\tau(U) \cap V \neq \emptyset$. A regular Hamiltonian level (H, e) is transitive if the Hamiltonian vector field X_H restricted to any energy hypersurface of $H^{-1}(\{e\})$ is transitive.*

It is well-known that if a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is such that $\mu_{\mathcal{E}_{H,e}}$ is *ergodic* then X_H is transitive on $\mathcal{E}_{H,e}$.

3.1.2 Transversal linear Poincaré flow and hyperbolicity

This section starts with the definition of the *transversal linear Poincaré flow*, which is based on the definition of linear Poincaré flow (Definition 2.2). After, we state some results using this flow.

Consider a Hamiltonian vector field X_H and a regular point x in M and let $e = H(x)$. Define $\mathcal{N}_x := N_x \cap T_x H^{-1}(\{e\})$, where $T_x H^{-1}(\{e\}) = Ker\, dH(x)$ is the tangent space to the energy level set. Thus, the $(dim(M) - 2)$-dimensional bundle $\mathcal{N}_x$ is $P_{X_H}^t(x)$-invariant

Definition 3.3 *The transversal linear Poincaré flow associated to H is given by*

$$\begin{aligned} \Phi_H^t(x) : \ \mathcal{N}_x &\to \mathcal{N}_{X_H^t(x)} \\ v &\mapsto \Pi_{X_H^t(x)} \circ D{X_H}_x^t(v), \end{aligned}$$

where $\Pi_{X_H^t(x)} : T_{X_H^t(x)}M \to \mathcal{N}_{X_H^t(x)}$ denotes the canonical orthogonal projection.

Observe that $\Phi_H^t(x) = P_H^t(x)|_{\mathcal{N}_x}$.

The proof of the following result can be found, for example, in [2].

Theorem 3.1 *Given a regular point $x \in \mathcal{E}_{H,e}$, then $\Phi^t_H(x)$ is a linear symplectomorphism for the symplectic form $\omega_{\mathcal{E}_{H,e}}$, that is, $\omega_{\mathcal{E}_{H,e}}(u,v) = \omega_{\mathcal{E}_{H,e}}(\Phi^t_H(x)\,u, \Phi^t_H(x)\,v)$, for any $u, v \in \mathcal{N}_x$.*

We recall that the set of symplectomorphisms forms a group under composition, denoted by $Sp(M,\omega)$, called *symplectic group*.

For any symplectomorphism, in particular for $\Phi^t_H(x)$, we have the following result.

Theorem 3.2 *(Symplectic eigenvalue theorem, [2]) Let $f \in Sp(M,\omega)$, $p \in M$ and σ an eigenvalue of $D_p f$ of multiplicity k. Then $1/\sigma$, $\bar{\sigma}$, $1/\bar{\sigma}$ are also eigenvalues of Df_p of multiplicity k. Moreover, the multiplicity of the eigenvalues $+1$ and -1, if they occur, is even.*

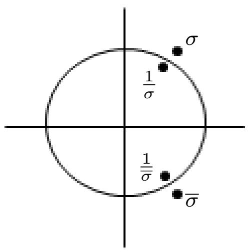

Figure 3.1: Spectrum of a symplectomorphism.

The following result is an extension of Lemma 2.2 to the symplectic framework (see [14, Lemma 2.3]).

Lemma 3.1 *Take a Hamiltonian $H \in C^2(M,\mathbb{R})$ and let Λ be a X^t_H-invariant, regular and compact subset of M. Then Λ is uniformly hyperbolic for X^t_H if and only if the induced transversal linear Poincaré flow Φ^t_H is uniformly hyperbolic on Λ.*

So, as explained in Section 2.1.2, we can define a *uniformly hyperbolic set* as follows.

Definition 3.4 *Let $H \in C^2(M,\mathbb{R})$. An X^t_H-invariant, compact and regular set $\Lambda \subset M$ is uniformly hyperbolic if $\mathcal{N}_\Lambda$ admits a Φ^t_H-invariant splitting $\mathcal{N}^s_\Lambda \oplus \mathcal{N}^u_\Lambda$ such that there is $\ell > 0$ satisfying*

$$\|\Phi^\ell_H(x)|_{\mathcal{N}^s_x}\| \le \frac{1}{2} \text{ and } \|\Phi^{-\ell}_H(X^\ell(x))|_{\mathcal{N}^u_{X^\ell(x)}}\| \le \frac{1}{2}, \text{ for any } x \in \Lambda.$$

Again, we remark that the constant $\frac{1}{2}$ can be replaced by any constant $\theta \in (0,1)$.

Now, we state the definition of *dominated splitting*, by using the transversal linear Poincaré flow.

Definition 3.5 *Take $H \in C^2(M,\mathbb{R})$ and let Λ be a compact, X_H^t-invariant and regular subset of M. Consider a Φ_H^t-invariant splitting $\mathcal{N} = \mathcal{N}^1 \oplus \cdots \oplus \mathcal{N}^k$ over Λ, for $1 \leq k \leq dim(M) - 2$, such that all the subbundles have constant dimension. This splitting is dominated if there exists $\ell > 0$ such that, for any $0 \leq i < j \leq k$,*

$$\|\Phi_H^\ell(x)|_{\mathcal{N}_x^i}\| \cdot \|\Phi_H^{-\ell}(X^\ell(x))|_{\mathcal{N}^j_{X^\ell(x)}}\| \leq \frac{1}{2}, \ \forall\, x \in \Lambda.$$

In the remaining of this section, we expose some results concerning on dominated splitting.

In the presence of a weakly hyperbolic closed orbit, the next two lemmas, due to Bessa and Dias, give us conditions to create a nearby elliptic closed orbit via a small perturbation.

Lemma 3.2 *([15, Proposition 3.2]) Let $H \in C^s(M^4,\mathbb{R})$, $2 \leq s \leq \infty$, and $\epsilon > 0$. There is $\theta > 0$ such that for any closed hyperbolic orbit Γ with period $\tau > 1$ and angle between $\mathcal{N}_q^u$ and $\mathcal{N}_q^s$ smaller than θ, for $q \in \Gamma$, there is $\tilde{H} \in C^\infty(M^4,\mathbb{R})$, ϵ-C^2-close to H, for which Γ is an elliptic closed orbit with period τ.*

Lemma 3.3 *([15, Proposition 3.3]) Let $H \in C^s(M^4,\mathbb{R})$, $2 \leq s \leq \infty$, $\epsilon > 0$ and $\theta > 0$. There exist positive constants ℓ and T, with $(T >> \ell)$, such that, if a hyperbolic closed orbit Γ with period $\tau > T$ has no ℓ-dominated splitting and is such that the angle between $\mathcal{N}_q^+$ and $\mathcal{N}_q^-$ is greater or equal than θ for all $q \in \Gamma$, then there exists $\tilde{H} \in C^\infty(M^4,\mathbb{R})$, ϵ-C^2-closed to H, for which Γ is an elliptic closed orbit with period τ.*

Conversely, the absence of elliptic periodic orbits for all nearby perturbations implies uniform bounds on the hyperbolic orbits with large enough period. This is an immediate consequence of the two previous lemmas.

Lemma 3.4 *Let $H \in C^s(M^4,\mathbb{R})$, for $2 \leq s \leq \infty$, and $\epsilon > 0$. Set $\theta = \theta(\epsilon, H)$, $\ell = \ell(\epsilon,\theta)$ and $T = T(\ell)$ given by Lemma 3.2 and Lemma 3.3. Assume that every*

Hamiltonian $\tilde{H}$, ϵ-C^2-close to H, do not admit elliptic closed orbits. Then, for every such $\tilde{H}$, any closed orbit with period larger that T is hyperbolic, ℓ-dominated and with angle between its stable and unstable directions bounded from bellow by θ.

Now, we present a result that we do not use directly. In fact, we appeal to techniques involved in its proof. Roughly speaking, the authors show in the proof that, for almost every point, either we have a dominated splitting, or else we can have Lyapunov exponents arbitrarily close to zero.

Theorem 3.3 *([14, Theorem 2]) There exists a C^2-dense subset $\mathcal{D}$ of $C^2(M^4,\mathbb{R})$ such that, if $H \in \mathcal{D}$ then there exists an invariant decomposition $M = D \cup Z$, unless a zero measure set, satisfying:*

- *$D = \cup_{n\in\mathbb{N}} D_{\ell_n}$, where D_{ℓ_n} is a set with ℓ_n-dominated splitting for Φ_H^t;*
- *X_H^t has zero Lyapunov exponents, for any point $p \in Z$.*

3.1.3 Topological dimension

The definition of *topological dimension* of a topological space X, denoted by $\dim(X)$, is not unique. However, on separable metrizable spaces all of them are equivalent. We state a well-known recursive definition of topological dimension that is is due, independently, to Menger and Urysohn (see [55, 76]) although its intuitive content goes back to Poincaré. In this formulation, the dimension of a space is the least integer d for which every point has arbitrarily small neighborhoods whose boundaries have dimension less than d.

Definition 3.6 *Let $d \geq 0$. We say that X satisfies $\dim(X) \leq d$ if there exists a basis of X made up of open sets whose boundaries have dimension less or equal than $d-1$. Also, we say that X has dimension d if $\dim(X) \leq d$ is true and $\dim(X) \leq d-1$ is false. Empty sets have dimension -1.*

The following result relates the topological dimension with the Lebesgue measure.

Theorem 3.4 *(Szpilrajn, [45]) Let $X \subset \mathbb{R}^d$ be a topological space. If X has zero Lebesgue measure then $\dim(X) < d$.*

3.1.4 Homoclinic classes

Given a hyperbolic closed orbit of saddle-type γ of a Hamiltonian H, with period π, and $p \in \gamma$. As in Definition 2.5, we define the *stable* and *unstable* manifolds of γ by

$$W_H^{s,u}(\gamma) = \bigcup_{0 \le t \le \pi} X_H^t(W_H^{s,u}(p)).$$

The *homoclinic class* of γ is defined by

$$\mathcal{H}_{\gamma,H} = \overline{W_H^s(\gamma) \pitchfork W_H^u(\gamma)},$$

where $\overline{S}$ stands for the closure of the set S and $\pitchfork$ denotes the transversal intersection of manifolds.

It is well-known that a non-empty homoclinic class is invariant by the flow, has a dense orbit, contains a dense set of closed orbits and is transitive. Moreover, the hyperbolic closed orbits of some index are dense in the homoclinic class (see [4], for example).

3.1.5 Resonance relations

Consider $H \in C^2(M, \mathbb{R})$ and recall that $dim(M) = 2d$. Let $\{\sigma_1, ..., \sigma_{2d}\}$ denote the set of eigenvalues of $DX_H(p)$, if $p \in Sing(H)$, or of $DX_H^\pi(q)$, if $q \in Per(H)$ has period π. A resonance relation between $\{\sigma_1, ..., \sigma_{2d}\}$ is an equality of the type

$$\sigma_i = \prod_{j=1}^{2d} \sigma_j^{k_j},$$

for some $i \in \{1, ..., 2d\}$ and some natural numbers $k_1, ..., k_{2d}$ such that either $k_i \neq 1$, or else there exists $j \neq i$ such that $k_j \neq 0$.

Since $\Phi_H^\pi(q)$ is a symplectomorphism, the following trivial resonance relations are satisfied:

$$\sigma_i = \sigma_i \prod_{k=1}^{d} (\sigma_k \sigma_{d+k})^{\alpha_k},$$

for naturals α_k. A resonance relation different from these ones is called a non-trivial resonance relation. Robinson proved in [69] that, C^2-generically, there are not non-trivial resonance relations.

Theorem 3.5 *[69, Theorem 1] There is a residual $\mathcal{R}$ in $C^2(M, \mathbb{R})$ such that, for any $H \in \mathcal{R}$, any $p \in Sing(H)$ and any $q \in Per(H)$ with period π, the eigenvalues of $DX_H(p)$ and of $DX_H^\pi(q)$ do not satisfy non-trivial resonance relations.*

We observe that, if we fix H in the previous residual set $\mathcal{R}$, sometimes we say that $Sing(H)$ and $Per(H)$ do not satisfy non-trivial resonances.

3.1.6 Pseudo-orbits

In this section we state the definition of *pseudo-orbit* for Hamiltonians, adapted from the one introduced by Bowen, in [27].

Definition 3.7 *Consider a Hamiltonian system* $(H, e, \mathcal{E}_{H,e})$ *and* $\epsilon > 0$. *A sequence* $\{x_i\}_{i=0}^n$ *on* $\mathcal{E}_{H,e}$, *with* $n \in \mathbb{N}$, *is an* ϵ*-pseudo-orbit on* $\mathcal{E}_{H,e}$ *if* $dist(X_H^1(x_i), x_{i+1}) < \epsilon$, *for any* $i \in \{0, ..., n-1\}$.

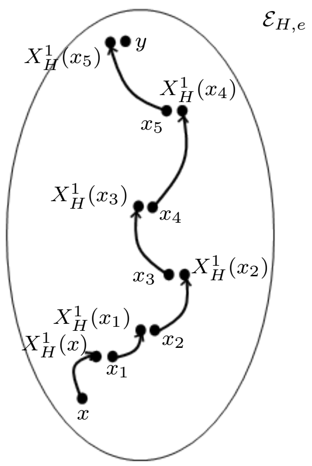

Figure 3.2: Representation of a pseudo-orbit on $\mathcal{E}_{H,e}$.

The *length* of the pseudo-orbit is equal to n.

Remark 10 *For divergence-free vector fields, and so for Hamiltonian vector fields, we have that* $\Omega(H|_{\mathcal{E}_{H,e}}) = \mathcal{E}_{H,e}$. *Therefore, any* $x, y \in \mathcal{E}_{H,e}$ *are connected by an* ϵ*-pseudo-orbit, for any* $\epsilon > 0$.

3.1.7 Lift axiom

Fix a regular point $p \in M$ and a small neighborhood U_p of p. By the Darboux Theorem (see, for example, [30, Theorem 1.18]), there is a smooth symplectic change of coordinates $\varphi_p : U_p \to T_pM$, such that $\varphi_p(p) = \vec{0}$. Denote by $N_{p,\delta}$ the ball centered in

$\vec{0}$ at the normal fiber at p and with radius $\delta > 0$. For a given $\delta > 0$ depending on p, let $f_H : \varphi_p^{-1}(N_{p,\delta}) \to \varphi^{-1}_{X_H^\tau(p)}(N_{X_H^\tau(p),1})$ be the canonical *Poincaré time-τ arrival map* associated to H, for $\tau > 0$. Note that if $p \in Per(H)$ has period π then we can chose any $0 < \tau < \pi$.

In [66], when proving the Closing Lemma for Hamiltonians, Pugh and Robinson show that the *lift axiom* is satisfied for Hamiltonians and they obtain the *closing* from the *lifting*. In rough terms, *lifting* is a way of pushing the orbits along a given direction by a small Hamiltonian perturbation, C^2-close to the identity. We never have to push in the direction of increasing energies. The key point on using the C^1 topology of X_H is that: "we can lift points p in prescribed directions v with results proportional to the support radius" ([66, pp. 266]).

Lift Axiom for Hamiltonians. Consider a Hamiltonian $H \in C^2(M, \mathbb{R})$ and let $\mathcal{U}$ be a C^2-neighborhood of H. Then there are $0 < \epsilon \leq 1$ and a continuous function $\delta : M \backslash Sing(H) \to (0,1)$, both depending on H and on $\mathcal{U}$, such that, for any $p \in M$ and $v \in N_{p,\delta(p)} \cap \varphi_p(H^{-1}(H(p)))$, there exists $\tilde{H} \in \mathcal{U}$ satisfying:

- $f_H^{-1} \circ f_{\tilde{H}}(p) = \varphi_p^{-1}(\epsilon v)$;
- $supp(X_{\tilde{H}} - X_H)$ is contained in the flowbox $\mathcal{T} = \bigcup_{t \in (0,T)} X_H^t(B_{\|v\|}(p))$, where $B_{\|v\|}(p)$ is taken in a transversal section of p and $T = T(y)$ is such that $T(p) = 1$ and $X_H^{T(y)}(y) \in B_{\|v\|}(X_H(p))$, for any $y \in B_{\|v\|}(p)$;
- if several such perturbations are made in disjoint flowboxes then their union-perturbation is also realizable by a Hamiltonian.

3.1.8 Perturbation flowboxes

Consider the standard cube $\mathbb{R}^{2d}$, tilled by smaller cubes by homotheties and translations. Given a symplectic chart $\varphi : U \to \mathbb{R}^{2d}$, for $U \subset \mathcal{E}_{H,e}$, the φ-pre-image of any tilled cube in $\varphi(U)$ is called *tiled cube of the chart* (U, φ) and it is denoted by $\mathcal{C}$. Note that $\mathcal{C} = \displaystyle\bigcup_{k=1}^{m} \mathcal{T}_k$, with $m \in \mathbb{N}$, where each $\mathcal{T}_k$ is called a *tile of* $\mathcal{C}$.

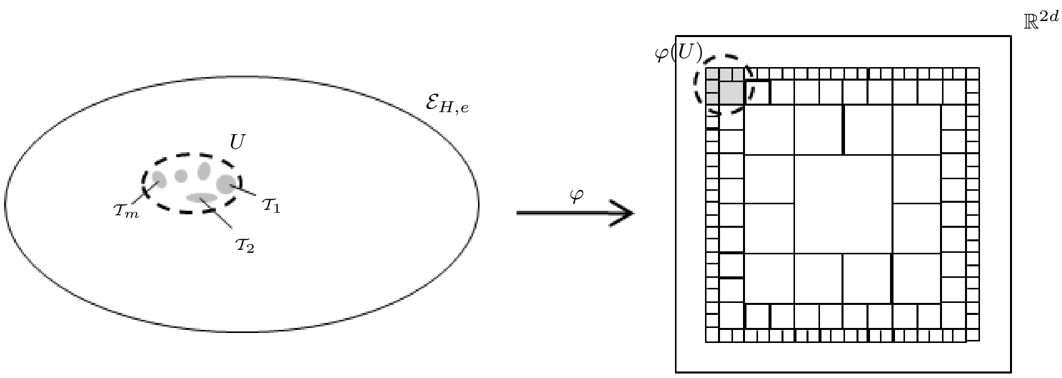

Figure 3.3: Representation of a tiled cube of the chart (U, φ).

Let us now state the definition of *pseudo-orbit preserving the tiling*.

Definition 3.8 *Consider a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$, a tiled cube of a chart $\mathcal{C} = \bigcup_{k=1}^{m} \mathcal{T}_k$ and a constant $T > 0$. We say that the pseudo-orbit $\{x_i\}_{i=0}^n$ on $\mathcal{E}_{H,e}$, with $n \in \mathbb{N}$,* preserves the tiling *in the injective flowbox*

$$\mathcal{F}_H(\mathcal{C}, T) = \bigcup_{t \in [0,T]} X_H^t(\mathcal{C})$$

if:

a) $x_0, x_n \notin \mathcal{F}_H(\mathcal{C}, T)$;

b) *for any $i \in \{1, ..., n-1\}$,*

- *if $x_i \in \mathcal{T}_k$ then $X_H^{-1}(x_{i+1}) \in \mathcal{T}_k$, for some $k \in \{1, ..., m\}$;*
- *if $x_i \in X_H^j(\mathcal{C})$ then $x_{i+1} = X_H^1(x_i)$, for some $j \in \{1, ..., T-1\}$.*

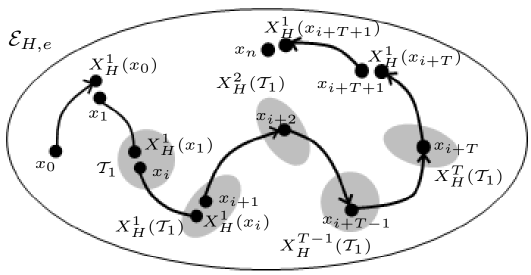

Figure 3.4: Representation of a pseudo-orbit preserving the tiling.

This definition asserts that the intersection of the pseudo-orbit $\{x_i\}_{i=0}^n$ with the flowbox $\mathcal{F}_H(\mathcal{C}, T)$ is an union of segments $\{x_j, ..., x_{j+T}\}$ such that $x_j \in \mathcal{C}$ and

$x_{j+k} = X_H^k(y_j)$, for every $k \in \{1, ..., T\}$, where y_j is a point in the same tile of x_j. Observe that if a pseudo-orbit preserves the tiling then we just have to take care about the jumps of the pseudo-orbit outside $\bigcup_{t\in[1,T-1]} X_H^t(\mathcal{C})$.

As Pugh and Robinson explained in [66], local perturbations on H do not change the energy hypersurfaces in the bottom and in the top of the flowboxes where the perturbations take place. So, we are allowed to push along the energy levels. This property motivates the following definition of *perturbation flowbox*.

Definition 3.9 *Fix a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$, $\epsilon > 0$ and an ϵ-C^2-neighborhood $\mathcal{U}$ of H. A tiled cube $\mathcal{C}$ is an ϵ-*perturbation flowbox of length T *for $(H,\mathcal{U})$ if, for any pseudo-orbit $\{x_i\}_{i=0}^n$ on $\mathcal{E}_{H,e}$ preserving the tiling in $\mathcal{F}_H(\mathcal{C}, T)$, there is $\tilde{H} \in \mathcal{U}$, such that $\tilde{H} = H$ outside $\mathcal{F}_H(\mathcal{C}, T-1)$, and a pseudo-orbit $\{y_j\}_{j=0}^m$ on $\mathcal{E}_{\tilde{H},e}$, with $m \in \mathbb{N}$, such that:*

- *$y_0 = x_0$ and $y_m = x_n$;*
- *$\tilde{H}(y_j) = e$, for any $j \in \{0, ..., m\}$;*
- *the intersection of the pseudo-orbit $\{y_j\}_{j=0}^m$ with $\mathcal{F}_H(\mathcal{C}, T)$ is an union of segments $\{y_i, ..., y_{i+T}\}$ such that $y_i \in \mathcal{C}$ and $y_{i+k} = X_{\tilde{H}}^k(y_i)$, for every $k \in \{1, ..., T\}$. Moreover, the segments of $\{y_j\}_{j=0}^m$ that do not intersect $\displaystyle\bigcup_{t\in[1,T-1]} X_H^t(\mathcal{C})$ are segments of the initial pseudo-orbit $\{x_i\}_{i=0}^n$, where the starting point belongs to $X_H^T(\mathcal{C})$ or coincides with x_0 and the ending point belongs to $\mathcal{C}$ or coincides with x_n.*

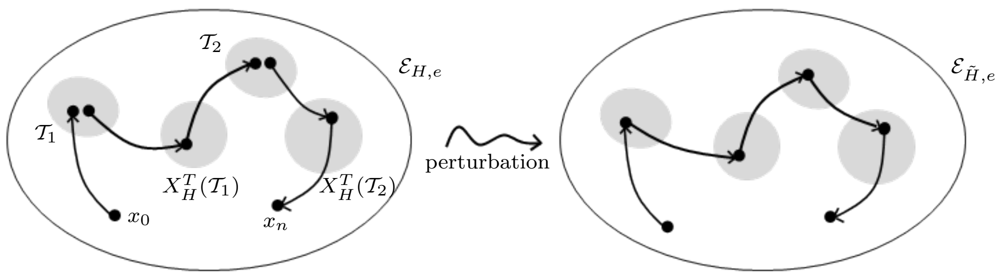

Figure 3.5: Perturbation in a tiled cube.

We call *support of a perturbation flowbox* $\mathcal{C}$, say $supp(\mathcal{C})$, to the union $\displaystyle\bigcup_{t\in[0,T]} X_H^t(\bar{\mathcal{C}})$.

The Hayashi Connecting Lemma is a key ingredient to prove the Connecting Lemma for pseudo-orbits of Hamiltonians (Lemma 1) and, as stated in [79], it can be adapted for Hamiltonians. From Definiton 3.9, we can extract a slightly stronger statement of the Connecting Lemma for Hamiltonians in [79, Theorem E], which can be seen as a theorem of existence of perturbation flowboxes.

Theorem 3.6 *Given a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ and $\epsilon > 0$, there exists $T > 0$ such that if any tiled cube $\mathcal{C}$ on $\mathcal{E}_{H,e}$ is a flowbox of length T then $\mathcal{C}$ is an ϵ-perturbation flowbox of length T.*

From the previous definitions and theorem, the following proposition follows immediately.

Proposition 3.1 *Consider a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ and let $\mathcal{U}$ be a C^2-neighborhood of H. For any pseudo-orbit $\{x_i\}_{i=0}^n$ on $\mathcal{E}_{H,e}$ preserving the tiling in a flowbox, there exist $\tilde{H} \in \mathcal{U}$ and $t > 0$, such that $\tilde{H}(x_0) = e$ and $X_{\tilde{H}}^t(x_0) = x_n$ on $\mathcal{E}_{\tilde{H},e}$.*

In fact, flowbox after flowbox, the Connecting Lemma for pseudo-orbits of Hamiltonians (Lemma 1) erases all the jumps of the pseudo-orbit. However, notice that the jumps of a pseudo-orbit have no reason to respect the tiling of some perturbation flowbox. To deal with this difficulty, we introduce the concept of *covering families* and of *avoidable closed orbits*.

3.1.9 Covering families

Given a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$, we want to cover the orbits on $\mathcal{E}_{H,e}$ by a family of perturbation flowboxes, with pairwise disjoint supports. Let $\mathcal{U}$ be a C^2-neighborhood of H and let $\mathcal{C}$ denote a family of perturbation flowboxes for $(H,\mathcal{U})$, with pairwise disjoint supports, and $\mathcal{V}$ denote a family of non-empty open subsets of $\mathcal{E}_{H,e}$ with pairwise disjoint supports.

Definition 3.10 *The family $\mathcal{C} = \bigcup_{k=1}^{m} \mathcal{T}_k$, for $m \in \mathbb{N}$, is a covering family of $\mathcal{E}_{H,e}$ if, for any $x \in \mathcal{E}_{H,e}$, there exist $t > 0$ and $1 \leq k \leq m$ such that $X_H^t(x) \in int(\mathcal{T}_k)$.*

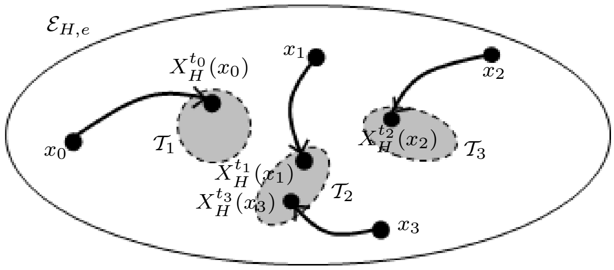

Figure 3.6: Representation of a covering family of $\mathcal{E}_{H,e}$.

In general, if $\mathcal{E}_{H,e}$ has closed orbits with small period then $\mathcal{E}_{H,e}$ has not a covering family. In fact, this kind of closed orbits is disjoint from the perturbation flowboxes. This motivates the definition of *covering families outside* $\mathcal{V} = \cup_{j=1}^{r} V_j$. The sets V_j $(1 \leq j \leq r)$ are, in fact, neighborhoods of these closed orbits with small period.

The following definition is an adaption of [8, Definition 3.2] for Hamiltonians.

Definition 3.11 *Fix a Hamiltonian system* $(H, e, \mathcal{E}_{H,e})$, $\epsilon > 0$ *and an* ϵ-C^2-*neighborhood* $\mathcal{U}$ *of* H. *A perturbation flowbox* $\mathcal{C}$ *for* $(H, \mathcal{U})$ *is a covering family of* $\mathcal{E}_{H,e}$ *outside* $\mathcal{V}$ *if there are*

- $t > 0$ *and* $\epsilon > 0$;
- *an open set* W_j *and a compact set* F_j, *such that* $F_j \subset W_j \subset V_j$, *for every* $j \in \{1, ..., r\}$;
- *a finite family of compacts* $\mathcal{D} = \bigcup_{i=1}^{s} D_i$ *on* $\mathcal{E}_{H,e}$, *such that every* D_i *is contained in the interior of a tile of* $\mathcal{C}$;
- *two parts* $\mathcal{D}_{a,j}$ *and* $\mathcal{D}_{o,j}$ *of* $\mathcal{D}$ *such that the support of the tiles of* $\mathcal{C}$ *containing this compacts is contained in* V_j, *for any* $j \in \{1, ..., r\}$,

such that

a) *any segment of any* ϵ-*pseudo-orbit on* $\mathcal{E}_{H,e}$ *with length greater or equal than* t *meets a compact* F_j *or a compact of* $\mathcal{D}$;

b) *any segment of any* ϵ-*pseudo-orbit on* $\mathcal{E}_{H,e}$ *starting outside* V_j *and ending inside* W_j *meets a compact of* $\mathcal{D}_{a,j}$, *for any* $j \in \{1, ..., r\}$;

c) *any segment of any ϵ-pseudo-orbit on $\mathcal{E}_{H,e}$ starting inside W_j and ending outside V_j meets a compact of $\mathcal{D}_{o,j}$, for any $j \in \{1, ..., r\}$;*

d) *for any $j \in \{1, ..., r\}$ and for any compact sets $D_a \subset \mathcal{D}_{a,j}$ and $D_o \subset \mathcal{D}_{o,j}$, there exists a pseudo-orbit with jumps inside the tiles of $\mathcal{C}$, with starting point inside D_a and ending point inside D_o.*

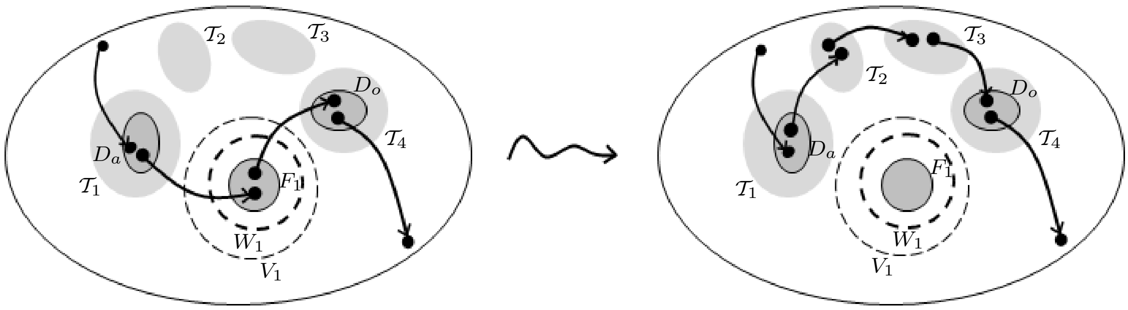

Figure 3.7: Covering family of $\mathcal{E}_{H,e}$ outside $\mathcal{V}$.

Roughly speaking, $\mathcal{C}$ is a *covering family of $\mathcal{E}_{H,e}$ outside $\mathcal{V}$* if any pseudo-orbit returns regularly to a compact $\mathcal{D} \subset int(\mathcal{T}_k)$, for some $1 \leq k \leq m$, during the time it passes out of $\mathcal{V}$. If the pseudo-orbit takes a long time to return to another compact set $\tilde{D} \subset \mathcal{D}$, it approaches some compacts $F_j \subset V_j$. For this, the pseudo-orbit must go through an *entrance compact* $D_a \subset \mathcal{D}$ and then through an *exit compact* $D_o \subset \mathcal{D}$. Moreover, we can even switch the segment of the pseudo-orbit between D_a and D_o by a pseudo-orbit with jumps inside the tiles of $\mathcal{C}$.

3.1.10 Avoidable closed orbits

Consider a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ and a closed orbit γ of H on $\mathcal{E}_{H,e}$. Let $\mathcal{U}$ be a C^2-neighborhood of H and fix $T > 0$ and $p \in \gamma$. The next definition is adapted from [8, Definition 3.10] for Hamiltonians.

Definition 3.12 *A closed orbit γ is avoidable for $(\mathcal{U}, T)$ if, for any neighborhood V_0 of γ and for any $t > 0$, there exist $\epsilon > 0$, open neighborhoods W and V of γ, such that $W \subset V \subset V_0$, and a perturbation flowbox $\mathcal{C}$ for $(H,\mathcal{U})$ of length T with disjoint supports, such that:*

a) *the support of $\mathcal{C}$ is contained in V;*

b) *there exist two families of compacts $\mathcal{D}_a$ and $\mathcal{D}_o$ contained in the interior of the tiles of $\mathcal{C}$ such that*

- *any segment of any ϵ-pseudo-orbit on $\mathcal{E}_{H,e}$ starting outside V and ending inside W has a point in a compact of $\mathcal{D}_a$;*
- *any segment of any ϵ-pseudo-orbit on $\mathcal{E}_{H,e}$ starting inside W and ending outside V has a point in a compact of $\mathcal{D}_o$;*

c) *for any compacts $D_a \in \mathcal{D}_a$ and $D_o \in \mathcal{D}_o$, there exist a pseudo-orbit on $\mathcal{E}_{H,e}$, with jumps inside the tiles of $\mathcal{C}$, starting in D_a and ending in D_o;*

d) *for any x in $\overline{\mathcal{C}}$, the time taking by $X_H^T(x)$ to return to $\overline{supp(\mathcal{C})}$ is bigger than t.*

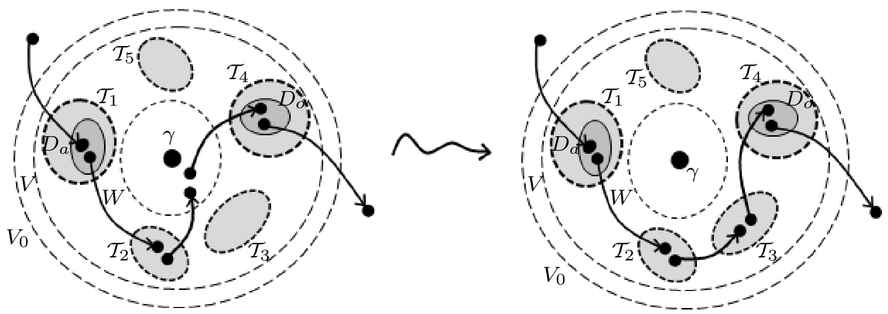

Figure 3.8: Representation of an avoidable closed orbit γ.

Therefore, a closed orbit γ is *avoidable for ($\mathcal{U}, T$)*, for fixed $T > 0$, if, for any $t > 0$, there exists a family $\mathcal{C}$ of perturbation flowboxes for $(H, \mathcal{U})$ of length T such that, given a pseudo-orbit with starting and ending points far from γ, but passing very close of γ, we can exchange the segments of the pseudo-orbit passing close of γ by segments of another pseudo-orbit with jumps inside the tiles $\mathcal{T}_k$ $(1 \leq k \leq m)$.

A closed orbit can be even characterized as *uniformly avoidable*.

Definition 3.13 *Let $(H, e, \mathcal{E}_{H,e})$ be a Hamiltonian system and $\mathcal{U}$ a C^2-neighborhood of H. The closed orbits of H on $\mathcal{E}_{H,e}$ are called uniformly avoidable if they are isolated and there exists a constant $T > 0$ such that any closed orbit of H on $\mathcal{E}_{H,e}$ is avoidable for ($\mathcal{U}, T$).*

This kind of orbits is used to derive *perturbation flowboxes* with disjoint supports, in such a way that the pseudo-orbits stay away from closed orbits with small period.

We anticipate that, if $\mathcal{E}_{H,e}$ has no orbits with small period and has all the closed orbits uniformly avoidable then we will be able to build a covering family of perturbation flowboxes for $\mathcal{E}_{H,e}$, as shown in Proposition 3.3, in Section 3.2.

3.1.11 C^2-perturbation results

In this section, we state some perturbation lemmas for the Hamiltonian setting, namely the *Closing Lemma*, the *Pasting Lemma* and the *Franks Lemma*.

The first perturbation result is a version of the *Closing Lemma* for Hamiltonians that we obtain by combining *Arnaud's Closing Lemma* (see [7]) with *Pugh and Robinson's Closing Lemma for Hamiltonians* (see [66]). It states that the orbit of a non-wandering point can be approximated, for a very long time, by a closed orbit of a nearby Hamiltonian.

Lemma 3.5 *Fix $H_1 \in C^2(M,\mathbb{R})$. Let $x \in M$ be a non-wandering point and ϵ, r and τ positive constants. Then, there exist $H_2 \in C^2(M,\mathbb{R})$, a closed orbit γ of H_2 with period π, $p \in \gamma$ and a map $g : [0,\tau] \to [0,\pi]$, close to the identity, such that:*

- *H_2 is ϵ-C^2-close to H_1;*
- *$dist\Big(X^t_{H_1}(x), X^{g(t)}_{H_2}(p)\Big) < r$, $0 \le t \le \tau$;*
- *$H_2 = H_1$ on $M \backslash A$ where $A = \bigcup_{0 \le t \le \tau}\Big(B_r\big(X^t_{H_1}(p)\big)\Big)$.*

The next lemma is a version of the C^1-Pasting Lemma ([6], Theorem 3.1) for Hamiltonians. Actually, in the Hamiltonian setting, the proof of this result is much more simple.

Lemma 3.6 *(Pasting Lemma for Hamiltonians) Fix $H_1 \in C^r(M,\mathbb{R})$, $2 \le r \le \infty$, and let K be a compact subset of M and U a small neighborhood of K. Given $\epsilon > 0$, there exists $\delta > 0$ such that if $H_2 \in C^s(M,\mathbb{R})$, for $2 \le s \le \infty$, is δ-$C^{min\{r,s\}}$-close to H_1 on U then there exist $H_3 \in C^s(M,\mathbb{R})$ and a closed set V such that:*

- *$K \subset V \subset U$;*
- *$H_3 = H_2$ on V;*

- $H_3 = H_1$ *on* U^c*;*
- H_3 *is* ϵ*-*$C^{min\{r,s\}}$*-close to* H_1*.*

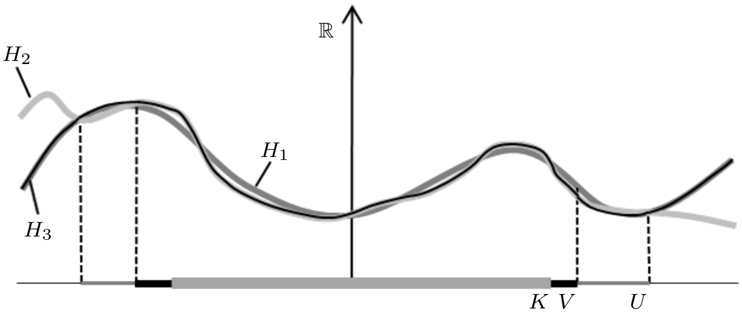

Figure 3.9: Perturbation given by the Pasting Lemma for Hamiltonians.

Proof: Consider $\{U_1, U_2\}$ an open cover of M, such that $U_1 := U$ and U_2 does not contain K. Then, there is a smooth partition of unity $\{\alpha_1, \alpha_2\}$, subordinate to $\{U_1, U_2\}$, such that $\alpha_i : M \to [0, 1]$ satisfies supp$(\alpha_i) \subseteq U_i$, for $i = 1, 2$, and $\alpha_1(x) + \alpha_2(x) = 1$, for any $x \in M$.

Letting $V := U_2^c$ and $H_3 := \alpha_1 H_2 + (1 - \alpha_1)H_1$, we have that:

- $K \subset V \subset U$;

- $H_3 = H_2$ on V, since $\alpha_1(x) = 1$ and $\alpha_2(x) = 0$, for any $x \in V$;

- $H_3 = H_1$ on U^c, since $\alpha_1(x) = 0$ and $\alpha_2(x) = 1$, for any $x \in U^c$;

- $\|H_3 - H_1\|_{C^{min\{r,s\}}} \leq \max\{\alpha_1(x)\} \, \|H_2 - H_1\|_{C^{min\{r,s\}}} = \|H_2 - H_1\|_{C^{min\{r,s\}}} < \delta$, since, by hypothesis, H_2 and H_1 are δ-$C^{min\{r,s\}}$-close. So, for $\delta > 0$ sufficiently small, we are done. $\square$

This result allows us to realize C^1-local perturbations in the Hamiltonian setting.

The last perturbation result, due to Vivier, is a version of Franks' Lemma for Hamiltonians (see [78]). Roughly speaking, it says that a perturbation of the transversal linear Poincaré flow can be realized as a linear Poincaré flow of a Hamiltonian.

Lemma 3.7 *(Vivier, [78]) Take* $H_1 \in C^2(M, \mathbb{R})$*,* $\epsilon > 0$*,* $\tau > 0$ *and* $x \in M$*. Then, there exists* $\delta > 0$ *such that for any flowbox* $\mathcal{F}(x)$ *of an injective arc of orbit* $X_{H_1}^{[0,t]}(x)$*, with* $t \geq \tau$*, and a transversal symplectic* δ*-perturbation* F *of* $\Phi_{H_1}^t(x)$*, there is* $H_2 \in C^2(M, \mathbb{R})$ *satisfying:*

- *H_2 is ϵ-C^2-close to H_1;*
- *$\Phi^t_{H_2}(x) = F$;*
- *$H_1 = H_2$ on $X^{[0,t]}_{H_1}(x) \cup (M \backslash \mathcal{F}(x))$.*

3.2 Proof of the Connecting Lemma for pseudo-orbits

This section contains the proof of the Connecting Lemma for pseudo-orbits of Hamiltonians.

Lemma 1 [Connecting Lemma for pseudo-orbits of Hamiltonians] Let (M, ω) denote a compact, symplectic $2d$-manifold, for $d \geq 2$. Take $H \in C^2(M, \mathbb{R})$ and a regular energy $e \in H(M)$, such that the eigenvalues of any closed orbit of H do not satisfy non-trivial resonances. Then, for any C^2-neighborhood $\mathcal{U}$ of H, for any energy hypersurface $\mathcal{E}_{H,e} \subset H^{-1}(\{e\})$ and for any $x, y \in \mathcal{E}_{H,e}$ connected by an ϵ-pseudo-orbit, for $\epsilon > 0$, there exist $\tilde{H} \in \mathcal{U}$ and $t > 0$ such that $e = \tilde{H}(x)$ and $X^t_{\tilde{H}}(x) = y$, on the analytic continuation $\mathcal{E}_{\tilde{H},e}$ of $\mathcal{E}_{H,e}$.

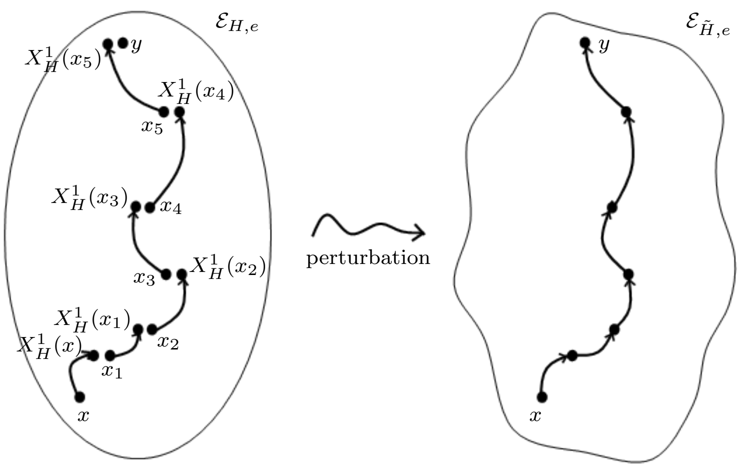

Figure 3.10: Perturbation given by the Connecting Lemma for pseudo-orbits.

As explained in [8, 24] and in [11], the proof of the *Connecting Lemma for pseudo-orbits* is splitted in three main parts. The first step to prove Lemma 1 concerns on *local perturbations*. These perturbations motivate the definition of *perturbation flowboxes* whose support must be in the interior of small open sets, pairwise disjoint till a sufficiently large number of iterates. Separately, we need to *analyze the dynamics near closed* *orbits with small period* because these orbits are not contained in any perturbation flowbox. Finally, we must analyze the *global dynamics*, in order to cover any orbit with perturbation flowboxes.

This strategy was firstly followed by Bonatti and Crovisier for diffeomorphisms (see [24]). Later, jointly with Arnaud (see [8]), these authors proceeded with this methodology to get the proof of the Connecting Lemma for pseudo-orbits of symplectomorphisms. The main novelties in the symplectomorphisms context are the need for the perturbations to be *symplectic* and also that the closed orbits can be *stably elliptic*. This means that the symplectomorphisms case cannot be reduced to the one treated in [24], where the closed orbits are assumed to be hyperbolic. That is why, in [8], the authors prove this result for symplectomorphisms, by doing the necessary changes.

For the Hamiltonian case, recall that the transversal linear Poincaré flow is, is fact, a symplectomorphism and observe that we are assuming the absence of singularities on the energy hypersurfaces. Keeping in mind the strategy described in [8], the novelties in the proof of the *Connecting Lemma for pseudo-orbits of Hamiltonian* are the statement of adequate definitions and, since the energy hypersurfaces are invariant by the Hamiltonian flow, the need for the pseudo-orbit being completely contained in the same energy hypersurface. Hence, we have to ensure the creation of symplectic perturbations without leaving the initial energy hypersurface. Recall that the energy hypersurface is indexed to the Hamiltonian. Thus, it may change when we perturb the Hamiltonian. That is why, in the statement of Lemma 1, we want the energy of the points in the pseudo-orbit to be kept constant, even if we C^2-perturb the Hamiltonian. However, since we are allowed to push along the energy levels (see [66, §9(a)]), the arguments stated in [8] can be adapted to the Hamiltonian case. At the end, we have a version of the Connecting Lemma for pseudo-orbits of Hamiltonians, where the condition on the persistence of the energy of the pseudo-orbit is trivially satisfied. Let us briefly explain how to prove Lemma 1.

Arnaud, Bonatti and Crovisier proved, in [8, Proposition 4.2], that if the eigenvalues of any closed orbit of a symplectomorphism do not satisfy non-trivial resonance relations, then the closed orbits are uniformly avoidable. Therefore, since the transversal linear Poincaré flow is a symplectomorphism, the following proposition follows directly for Hamiltonians.

Proposition 3.2 *Consider a Hamiltonian $H \in C^2(M, \mathbb{R})$. If, for any closed orbit p of H with period π, the eigenvalues of $\Phi_H^{\pi}(p)$ do not satisfy non-trivial resonances then the closed orbits of H are uniformly avoidable.*

As explained before, to prove this proposition, the authors take into account that the closed orbits can be *hyperbolic* (case analyzed in [24]) but also *completely elliptic* or *elliptic* (see Definition 2.1 for more details).

Observe that, by the previous proposition, Theorem 3.5 implies that the closed orbits of a C^2-generic Hamiltonian are uniformly avoidable.

Now, by Proposition 3.2, to prove the Connecting Lemma for pseudo-orbits of Hamiltonians it is enough to show the following result.

Theorem 3.7 *Consider a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ such that the closed orbits of H on $\mathcal{E}_{H,e}$ are uniformly avoidable. Then, for any C^2-neighborhood $\mathcal{U}$ of H and for any $x, y \in \mathcal{E}_{H,e}$, there is $\tilde{H} \in \mathcal{U}$ and $t > 0$, such that $\tilde{H}(x) = e$ and $X_{\tilde{H}}^t(x) = y$, on the analytic continuation $\mathcal{E}_{\tilde{H},e}$ of $\mathcal{E}_{H,e}$.*

It is obvious that Theorem 3.7 follows immediately if $y \in \mathcal{O}_H(x)$. In fact, to prove Lemma 1, it is enough to show Theorem 3.7 for some kind of points $x, y \in \mathcal{E}_{H,e}$.

Lemma 3.8 *([8, Lemma 3.12]) Consider a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ such that the closed orbits on $\mathcal{E}_{H,e}$ are isolated. Take any $x, y \in \mathcal{E}_{H,e}$ such that $y \notin \mathcal{O}_H(x)$. Then, there exist $\tilde{x}$ and $\tilde{y}$, arbitrarily close to x and y, such that either $\tilde{y} \in \mathcal{O}_H(\tilde{x})$, or else $\tilde{x}$ and $\tilde{y}$ are not closed orbits.*

Recall that a uniformly avoidable closed orbit is indeed isolated. So, by the previous lemma, the proof of Lemma 1 is reduced to the proof of Theorem 3.7, when x, y are not closed orbits. In fact, if $y \notin \mathcal{O}_H(x)$ and x or y are closed orbits, we just have to apply Theorem 3.7 to $\tilde{x}$ and $\tilde{y}$, given by Lemma 3.8. Then, a Hamiltonian perturbation of the identity sends x, y into $\tilde{x}, \tilde{y}$, and it allows us to conclude the result for any x and y in $\mathcal{E}_{H,e}$.

Recall that H satisfies the *lift axiom* and that any two distinct points $x, y \in \mathcal{E}_{H,e}$ are connected by an ϵ-pseudo-orbit, for any $\epsilon > 0$. Therefore, by Lemma 3.8, we can reduce the proof of Theorem 3.7, and so of the Connecting Lemma for pseudo-orbits of Hamiltonians, to the proof of Proposition 3.3 and Proposition 3.4 bellow.

Proposition 3.3 *Take a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$, such that H satisfies the lift axiom and any closed orbit of H on $\mathcal{E}_{H,e}$ is uniformly avoidable. Let $\mathcal{U}_0$ be a C^2-neighborhood of H and $x, y \in \mathcal{E}_{H,e}$ be such that $x, y \notin Per(H)$ and $y \notin \mathcal{O}_H(x)$. Then there exist a neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of H, a family of disjoint open sets $\mathcal{V}$ and a family of perturbation flowboxes $\mathcal{C}$ for $(H, \mathcal{U})$ with disjoint supports, both $\mathcal{V}$ and $\mathcal{C}$ not containing x nor y, such that $\mathcal{C}$ is covering $\mathcal{E}_{H,e}$ outside $\mathcal{V}$.*

In this case, we want to build a family of perturbation flowboxes in a neighborhood of closed orbits. Let us sketch the proof of this proposition, adapting the ideas of the proof in [8, Proposition 3.13].

We want to construct finitely many disjoint perturbation flowboxes, whose union meets every orbit of $\mathcal{E}_{H,e}$, called *topological tower* of order T. Clearly, the existence of closed orbits with small period, even in a finite number, goes against the existence of a topological tower. However, if we construct a perturbation flowbox $\mathcal{C}$, covering $\mathcal{E}_{H,e}$ outside a finite family of disjoint open sets $\mathcal{V} = \cup_{i=1}^{j} V_i$, we can include any closed orbit with small period in the interior of some V_i. In this case, we have a finite family of disjoint perturbation flowboxes $\mathcal{C}$ far from closed orbits with small period. Now, it remains to show how can we build these disjoint perturbation flowboxes with length T.

Remark 11 *We state the definition of a flow, built under a ceiling function h. Consider a measure space Σ, a map $R : \Sigma \to \Sigma$, a measure $\tilde{\mu}$ in Σ and an integrable function $h : \Sigma \to [c, +\infty]$, with $c > 0$ and $\int_\Sigma h(x) d\tilde{\mu}(x) = 1$. The flow*

$$\begin{aligned} S^s : \Sigma \times \mathbb{R} &\longrightarrow \Sigma \times \mathbb{R} \\ (x, r) &\mapsto \left(R^k(x), r + s - \sum_{i=0}^{k-1} h(R^i(x)) \right), \end{aligned}$$

where $k \in \mathbb{Z}$ is uniquely defined by $\displaystyle\sum_{i=0}^{k-1} h(R^i(x)) \leq r + s < \sum_{i=0}^{k} h(R^i(x))$, is called a special flow. In fact, the flow S^s moves the point (x, r) to $(x, r + s)$ at velocity one, until it hits the graph of h. After this, the point returns to Σ and continues its journey.

The Ambrose-Kakutani's Theorem states that a flow having the set of critical points with zero Lebesgue measure is isomorphic to a special flow (see [3]).

Recall that any closed orbit of H on the regular energy hypersurface $\mathcal{E}_{H,e}$ is uniformly avoidable, and so isolated. Then, H has a finite number of closed orbits with small period. Therefore, by Ambrose-Kakutani's Theorem in [3], φ_H^t is equivalent to a special flow. Now, following [12, Section 3.6.1], with the obvious changes, we can build a topological tower with very high towers in order to have enough time to perform a lot of small non-overlapped perturbations.

The next proposition, jointly with Proposition 3.3, finishes the proof of Lemma 1.

Proposition 3.4 *Consider a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ and a neighborhood $\mathcal{U}$ of H. Let $\mathcal{C}$ denote a family of perturbation flowboxes for $(H,\mathcal{U})$ covering $\mathcal{E}_{H,e}$ outside a family of open sets $\mathcal{V}$. Take any $x, y \in \mathcal{E}_{H,e}$ outside the support of $\mathcal{C}$ and outside of any $V \in \mathcal{V}$. Then there exist $\tilde{H} \in \mathcal{U}$ and $t > 0$, such that $\tilde{H}(x) = e$ and $X_{\tilde{H}}^t(x) = y$, on the analytic continuation $\mathcal{E}_{\tilde{H},e}$ of $\mathcal{E}_{H,e}$.*

By Proposition 3.1, if the hypothesis of the previous proposition ensure that a pseudo-orbit connecting x and y preserves the tiling of $\mathcal{C}$, then we are done. In fact, as explained in Section 3.1.9, given that the perturbation flowbox $\mathcal{C}$ covers $\mathcal{E}_{H,e}$ outside $\mathcal{V}$, every orbit on $\mathcal{E}_{H,e}$ spends a uniformly bounded time to return to the interior of any tile of $\mathcal{C}$. It is straightforward to see that the same holds for any ϵ-pseudo-orbit, with small $\epsilon > 0$. Moreover, if we choose $\epsilon > 0$ even smaller, we can modify the pseudo-orbit in such a way that, whenever the pseudo-orbit returns to the interior of some tile, we add at this time all the next jumps of the pseudo-orbit until the next return to a tile, defining, in this way, a new jump. The final jump respects the tile and is small, because the number of grouped jumps is uniformly bounded. In this way, we construct a pseudo-orbit preserving the tiling of $\mathcal{C}$.

3.3 Proof of the Hamiltonian results

This section includes the proof of Theorem 2, Theorem 3, Theorem 4, Theorem 6, Corollary 2, Corollary 4 and Theorem 10.

3.3.1 Openness and structural stability

Following classic arguments of hyperbolic dynamics, in this section, we prove the *openness* and the *structural stability* of Anosov Hamiltonian systems defined on any even-dimensional symplectic manifold (see, for example, [29, 46]). For this, the continuity of hyperbolic sets plays an important role.

Let us start with the definition of α-*cones*. Consider a Hamiltonian $H \in C^2(M, \mathbb{R})$ and let Λ be a regular, X_H^t-invariant and uniformly hyperbolic subset of M with decomposition $\mathcal{N}_\Lambda = \mathcal{N}_\Lambda^- \oplus \mathcal{N}_\Lambda^+$. Since the subbundles $\mathcal{N}^-$ and $\mathcal{N}^+$ are continuous, we extend them to continuous subbundles $\tilde{\mathcal{N}}^-$ and $\tilde{\mathcal{N}}^+$, defined on a regular neighborhood $\mathcal{U}$ of Λ. Fix $x \in \mathcal{U}$ and $v \in \mathcal{N}_x$ and let $v = v^- + v^+$, with $v^- \in \mathcal{N}_x^-$ and $v^+ \in \mathcal{N}_x^+$. For $\alpha > 0$, define the *stable* and *unstable cones* of size α by

$$K_\alpha^-(x) = \left\{v \in \mathcal{N}_x : \left\|v^+\right\| \leq \alpha \left\|v^-\right\|\right\},$$

$$K_\alpha^+(x) = \left\{v \in \mathcal{N}_x : \left\|v^-\right\| \leq \alpha \left\|v^+\right\|\right\}.$$

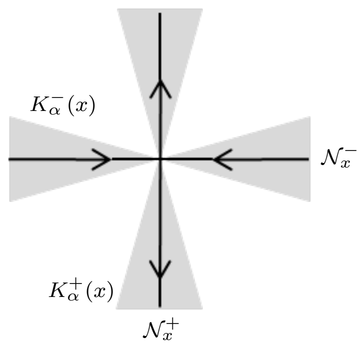

Figure 3.11: Representation of the stable and unstable cones.

Now, we prove the following standard proposition.

Proposition 3.5 *Consider $H \in C^2(M, \mathbb{R})$ and $\Lambda \subset M$ a compact, regular and X_H^t-invariant set. Suppose that there are $m \in \mathbb{N}$, $\alpha > 0$ and continuous subspaces $\tilde{\mathcal{N}}_x^-$ and $\tilde{\mathcal{N}}_x^+$, for every $x \in \Lambda$, such that $\mathcal{N}_x = \tilde{\mathcal{N}}_x^- \oplus \tilde{\mathcal{N}}_x^+$, and that the α-cones $K_\alpha^-(x)$ and $K_\alpha^+(x)$, determined by the subspaces, satisfy*

- $\Phi_H^t(x)\big(K_\alpha^+(x)\big) \subset K_\alpha^+\big(X_H^t(x)\big)$, *for $t \geq 0$;*
- $\Phi_H^{-t}\big(X_H^t(x)\big)\big(K_\alpha^-\big(X_H^t(x)\big)\big) \subset K_\alpha^-(x)$, *for $t \geq 0$;*

- $\|\Phi_H^m(x)v\| < \|v\|$, *for any* $v \in K_\alpha^-(x)\setminus\{0\}$;
- $\left\|\Phi_H^{-m}(x)v\right\| < \|v\|$, *for any* $v \in K_\alpha^+(x)\setminus\{0\}$.

Then Λ *is a uniformly hyperbolic set.*

Proof: By compactness of Λ and of the unit tangent bundle of M, there is a constant $\theta \in (0,1)$ such that $\|\Phi_H^m(x)v\| \leq \theta\,\|v\|$, for any $v \in K_\alpha^-(x)$ and $\left\|\Phi_H^{-m}(x)v\right\| \leq \theta\,\|v\|$, for any $v \in K_\alpha^+(x)$.

Now, for any $x \in \Lambda$, define

$$\mathcal{N}_x^- := \bigcap_{n\in\mathbb{N}_0} \Phi_H^{-n}\big(X_H^n(x)\big)K_\alpha^-\big(X_H^n(x)\big) \qquad \text{and}$$

$$\mathcal{N}_x^+ := \bigcap_{n\in\mathbb{N}_0} \Phi_H^{n}\big(X_H^{-n}(x)\big)K_\alpha^+\big(X_H^{-n}(x)\big).$$

Obviously, we have that $\mathcal{N}_x = \mathcal{N}_x^- \oplus \mathcal{N}_x^+$ and that the fibers are Φ_H^t-invariant. Also, observe that $\mathcal{N}_x^- \subset K_\alpha^-(x)$ and $\mathcal{N}_x^+ \subset K_\alpha^+(x)$. So, $\|\Phi_H^m(x)v\| \leq \theta\,\|v\|$, for any $v \in \mathcal{N}_x^-$ and $\left\|\Phi_H^{-m}(x)v\right\| \leq \theta\,\|v\|$, for any $v \in \mathcal{N}_x^+$. Thus, by Definition 3.4, Λ is a uniformly hyperbolic set. $\square$

Now, we prove the openness of the set $\mathcal{A}_\omega^2(M)$.

Theorem 2 ([13, Theorem 3]) The set $\mathcal{A}_\omega^2(M^{2d})$ is open, for $d \geq 2$.

Proof: The proof of the openness follows standard cone-fields arguments that can be found, for instance, in the book of Brin and Stuck (see [29]).

Fix $d \geq 2$. According to Definition 1.7, we want to prove that, given a Hamiltonian system $(H, e, \mathcal{E}_{H,e}) \in \mathcal{A}_\omega^2(M^{2d})$, there exist a C^2-neighborhood $\mathcal{U}$ of H and $\epsilon > 0$ such that, for any $\tilde{H}$ in $\mathcal{U}$ and any $\tilde{e} \in (e-\epsilon, e+\epsilon)$, the Hamiltonian system $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}})$ is also Anosov.

Assume that $(H, e, \mathcal{E}_{H,e}) \in \mathcal{A}_\omega^2(M^{2d})$. Since $(H, e, \mathcal{E}_{H,e})$ is Anosov, we have that $\mathcal{E}_{H,e}$ is uniformly hyperbolic and that $\mathcal{N}_{\mathcal{E}_{H,e}}$ admits the Φ_H^t-invariant and hyperbolic splitting

$$\mathcal{N}_{\mathcal{E}_{H,e}} = \mathcal{N}_{\mathcal{E}_{H,e}}^- \oplus \mathcal{N}_{\mathcal{E}_{H,e}}^+.$$

Since $\mathcal{E}_{H,e}$ is regular, its analytic continuation $\mathcal{E}_{\tilde{H},\tilde{e}}$ is well-defined over a small neighborhood $\mathcal{W}$ of $\mathcal{E}_{H,e}$. Then, we continuously extend $\mathcal{N}^-$ and $\mathcal{N}^+$ over $\mathcal{E}_{H,e}$ to $\tilde{\mathcal{N}}^-$ and $\tilde{\mathcal{N}}^+$

over $\mathcal{W}$. Choosing $\alpha > 0$ and $\mathcal{W}$ small enough, for any $\mathcal{E}_{\tilde{H},\tilde{e}} \in \mathcal{W}$, the stable and unstable α-cones, determined by $\tilde{\mathcal{N}}^-$ and $\tilde{\mathcal{N}}^+$, satisfy the assumptions of Proposition 3.5 for $\Phi^t_{\tilde{H}}$ on $\mathcal{E}_{\tilde{H},\tilde{e}}$. This means that $\mathcal{E}_{\tilde{H},\tilde{e}}$ is uniformly hyperbolic. So, the Hamiltonian system $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}})$ is Anosov, for any $\tilde{H} \in \mathcal{U}$ and any $\tilde{e} \in (e-\epsilon, e+\epsilon)$. $\square$

We end this section with the proof of the structural stability of Anosov Hamiltonian systems.

Theorem 3 ([13, Theorem 3]) The elements of $\mathcal{A}^2_\omega(M^{2d})$ are C^2-structurally stable, for $d \geq 2$.

Proof: Fixing $d \geq 2$ and $(H, e, \mathcal{E}_{H,e}) \in \mathcal{A}^2_\omega(M^{2d})$, we have that $\mathcal{E}_{H,e}$ is uniformly hyperbolic and regular. So, the measure $\mu_{\mathcal{E}_{H,e}}$ is well-defined and is preserved by the flow $X^t_H|_{\mathcal{E}_{H,e}}$ (see Section 3.1.1). Hence, by the *Anosov Theorem* (see [5]), $\mu_{\mathcal{E}_{H,e}}$ is ergodic.

Now, taking an arbitrarily small neighborhood $\mathcal{W}$ of $\mathcal{E}_{H,e}$, there exist a C^2-neighborhood $\mathcal{U}$ of H and $\epsilon > 0$ such that, for any $\tilde{H} \in \mathcal{U}$ and any $\tilde{e} \in (e-\epsilon, e+\epsilon)$, the analytic continuation $\mathcal{E}_{\tilde{H},\tilde{e}}$ is well-defined. There is $\eta > 0$ such that, for any $\tilde{H} \in \mathcal{U}$, η-C^2-close to H, and any $\delta > 0$, there is a compact, $X^t_{\tilde{H}}$-invariant and hyperbolic set $\tilde{\Lambda}$ and a homeomorphism $h\colon \mathcal{E}_{H,e} \to \tilde{\Lambda}$, with $dist(id, h) + dist(id, h^{-1}) < \delta$, that maps orbits of X^t_H to orbits of $X^t_{\tilde{H}}$, preserving their orientation (see, for example, [46, Theorem 18.2.3]).

Now, it is enough to prove that $\tilde{\Lambda} = \mathcal{E}_{\tilde{H},\tilde{e}}$. By compactness, $\mathcal{E}_{H,e}$ has a dense orbit and so, since h takes orbits into orbits, there is also a dense orbit in $\tilde{\Lambda}$. Hence, densely, the $\tilde{H}$-image of the points in $\tilde{\Lambda}$ is constant. Now, extending to the closure, we conclude that there exists $\tilde{e} \in (e-\epsilon, e+\epsilon)$ such that $\tilde{\Lambda} \subset \mathcal{E}_{\tilde{H},\tilde{e}}$. By the openness of Anosov Hamiltonian systems, we have that $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}})$ is still Anosov and so, by *Anosov's theorem*, $\mu_{\mathcal{E}_{\tilde{H},\tilde{e}}}$ is ergodic. Thus, once $\tilde{\Lambda} \subset \mathcal{E}_{\tilde{H},\tilde{e}}$ is compact and $X^t_{\tilde{H}}$-invariant, we must have $\mu_{\mathcal{E}_{\tilde{H},\tilde{e}}}(\tilde{\Lambda}) = 0$, or else $\mu_{\mathcal{E}_{\tilde{H},\tilde{e}}}(\tilde{\Lambda}) = \mu_{\mathcal{E}_{\tilde{H},\tilde{e}}}(\mathcal{E}_{\tilde{H},\tilde{e}})$. If $\mu_{\mathcal{E}_{\tilde{H},\tilde{e}}}(\tilde{\Lambda}) = 0$ then, by Theorem 3.4, $\dim(\tilde{\Lambda}) < 2d-1$. However, this is not possible because, given that the homeomorphism h preserves the topological dimension, $\dim(\tilde{\Lambda}) = \dim(\mathcal{E}_{H,e}) = 2d-1$. Therefore, $\mu_{\mathcal{E}_{\tilde{H},\tilde{e}}}(\tilde{\Lambda}) = \mu_{\mathcal{E}_{\tilde{H},\tilde{e}}}(\mathcal{E}_{\tilde{H},\tilde{e}})$ and, by compactness, we have that $\tilde{\Lambda} = \mathcal{E}_{\tilde{H},\tilde{e}}$.

Hence, there is a homeomorphism from $\mathcal{E}_{H,e}$ to $\mathcal{E}_{\tilde{H},\tilde{e}}$, preserving orbits and their orientations. This means that $(H, e, \mathcal{E}_{H,e}) \in \mathcal{A}^2_\omega(M^{2d})$ is C^2-structurally stable, for any $d \geq 2$. $\square$

Remark 12 *Since the homeomorphism h can be chosen arbitrarily close to the identity, we proved, in fact, that Anosov Hamiltonian systems are strongly C^2-structurally stable.*

3.3.2 Star property and uniform hyperbolicity

In this section, we show that a Hamiltonian star system, defined on a 4-dimensional symplectic manifold, is an Anosov Hamiltonian system.

Theorem 4 ([13, Theorem 1]) If $(H, e, \mathcal{E}^\star_{H,e}) \in \mathcal{G}^2_\omega(M^4)$ then $(H, e, \mathcal{E}^\star_{H,e}) \in \mathcal{A}^2_\omega(M^4)$.

The proof of this result is splitted into two lemmas. The first lemma deals with conditions that assure the existence of a dominated splitting on a given energy hypersurface (see Lemma 3.9). After that, in Lemma 3.10, we show how to derive uniform hyperbolicity from the existence of a dominated splitting in the 4-dimensional Hamiltonian setting.

We observe that, whenever $dim(\mathcal{E}_{H,e}) = 3$, $\mathcal{E}_{H,e}$ is called an *energy surface* instead of energy hypersurface.

Lemma 3.9 *If $(H, e, \mathcal{E}^\star_{H,e}) \in \mathcal{G}^2_\omega(M^4)$ then Φ^t_H admits a dominated splitting over $\mathcal{E}^\star_{H,e}$.*

Proof: Fix $(H, e, \mathcal{E}^\star_{H,e}) \in \mathcal{G}^2_\omega(M^4)$. Then there exist a C^2-neighborhood $\mathcal{U}$ of H and $\epsilon > 0$ such that, for any $\tilde{H} \in \mathcal{U}$ and any $\tilde{e} \in (e - \epsilon, e + \epsilon)$, the analytic continuation $\mathcal{E}_{\tilde{H},\tilde{e}}$ of $\mathcal{E}_{H,e}$ also has all the closed orbits hyperbolic. Observe that, since $\mathcal{E}^\star_{H,e}$ is regular, the invariant volume measure $\mu_{\mathcal{E}^\star_{H,e}}$ is well-defined on $\mathcal{E}^\star_{H,e}$ (see Section 3.1.1).

By contradiction, assume that Φ^t_H does not admit a dominated splitting over $\mathcal{E}^\star_{H,e}$. Then there exist a $\mu_{\mathcal{E}^\star_{H,e}}$-positive measure and X^t_H-invariant set $B \subset \mathcal{E}^\star_{H,e}$ such that B does not admit a dominated splitting for Φ^t_H. In this case we claim that

Claim 3.1 *For any $\ell \in \mathbb{N}$, there exists a $\mu_{\mathcal{E}^\star_{H,e}}$-positive measure and X^t_H-invariant subset of B, say Γ_ℓ, such that Γ_ℓ does not admit an ℓ-dominated splitting for Φ^t_H.*

If this claim is not true, there exists $\ell \in \mathbb{N}$ such that any Γ_ℓ, in the above conditions, admits an ℓ-dominated splitting for Φ^t_H. But, taking $\Gamma_\ell := B$, we reach a contradiction, since B does not admit a dominated splitting for Φ^t_H.

By hypothesis, given $\epsilon > 0$, any Hamiltonian $\tilde{H} \in \mathcal{U}$, ϵ-C^2-close to H, has no elliptic closed orbits. Then, by Lemma 3.4, for every such a Hamiltonian $\tilde{H}$, there are constants $\theta = \theta(\epsilon, H)$, $\ell = \ell(\epsilon, \theta)$ and $T = T(\ell)$ such that any closed orbit with period larger than T is ℓ-dominated and the angle between its stable and unstable directions is bounded from below by θ. Notice that these closed orbits are all hyperbolic.

Since $\mathcal{E}^\star_{H,e}$ is a compact energy surface and $\mu_{\mathcal{E}^\star_{H,e}}$ is X^t_H-invariant, we can apply the *Poincaré Recurrence Theorem* on $\mathcal{E}^\star_{H,e}$. Let R be a measurable subset of Γ_ℓ with $\mu_{\mathcal{E}^\star_{H,e}}$-total measure in Γ_ℓ, given by the *Poincaré Recurrence Theorem* with respect to $X_H|_{\mathcal{E}^\star_{H,e}}$. Then, $\mu_{\mathcal{E}^\star_{H,e}}(R) = \mu_{\mathcal{E}^\star_{H,e}}(\Gamma_\ell)$.

We observe that the set of closed orbits with period less than $k \in \mathbb{N}$ is a set of zero measure. Let Q denote the subset of points of Γ_ℓ having zero Lyapunov exponents for X_H on $\mathcal{E}^\star_{H,e}$. We want to choose a point $x \in Q \cap R$. If $\mu_{\mathcal{E}^\star_{H,e}}(Q) > 0$, we are done. Now, let us consider the reverse case. Assume that $\mu_{\mathcal{E}^\star_{H,e}}$-a.e. point x in Γ_ℓ has a nonzero Lyapunov exponent for $X_H|_{\mathcal{E}^\star_{H,e}}$, that is, $\mu_{\mathcal{E}^\star_{H,e}}(Q) = 0$. In this case, the idea is to choose $x \in R$ and use the techniques involved in the proof of Theorem 2, in [14], in order to force the decay of the Lyapunov exponents. So, for ℓ sufficiently large and $\eta > 0$ arbitrarily small, there exist $T_0 > 0$ and $H_1 \in \mathcal{U}$, C^2-close to H, such that x has Lyapunov exponents less than η for $X_{H_1}|_{\mathcal{E}^\star_{H_1,H_1(x)}}$, that is,

$$\exp(-\eta t) < \left\|\Phi^t_{H_1}(x)\right\| < \exp(\eta t), \text{ for any } t > T_0.$$

Now, fixing $\delta \in \left(0, \frac{\log 2}{2\ell}\right)$ and $\eta < \delta$, there is $T_x \in \mathbb{R}$ such that

$$\exp(-\delta t) < \left\|\Phi^t_{H_1}(x)\right\| < \exp(\delta t), \text{ for any } t \geq T_x.$$

Notice that we can assume $T_x \geq T$. Given that $x \in R$, we can apply the Closing Lemma for Hamiltonians (Lemma 3.5) and conclude that the $X^t_{H_1}$-orbit of x can be approximated, for a very long recurrent time $\tilde{T} > T_x$, by a closed orbit of a C^1-close flow $X^t_{H_2}$: given $r, \tilde{T} > 0$, we can find $H_2 \in \mathcal{U}$, C^2-close to H_1, a closed orbit Γ of H_2 with period π as large as we want, $\hat{T} > \tilde{T}$ and $g : \left[0, \tilde{T}\right] \to [0, \pi]$, close to the identity, such that, for $p \in \Gamma$ in $\mathcal{E}^\star_{H_2,H_2(p)}$,

- $dist\Big(X^t_{H_1}(x), X^{g(t)}_{H_2}(p)\Big) < r$, for $0 \leq t \leq \hat{T}$;

- $H_1 = H_2$ on $M \backslash \bigcup_{0 \le t \le \hat{T}} \Big(B_r\big(X^t_{H_1}(x)\big)\Big)$.

Letting r be small enough, we also have that

$$\exp(-\delta\pi) < \left\|\Phi^{\pi}_{H_2}(p)\right\| < \exp(\delta\pi), \tag{3.1}$$

where $\pi > T$. Since, by construction, $H_2 \in \mathcal{U}$ and $\pi > T$, by Lemma 3.4, we have that

$$\left\|\Phi^{\ell}_{H_2}(q)|_{\mathcal{N}^-_q}\right\| \le \frac{1}{2}\left\|\Phi^{\ell}_{H_2}(q)|_{\mathcal{N}^+_q}\right\|,$$

for every q in the $X^t_{H_2}$-orbit of p. Define $p_i = X^{i\ell}_{H_2}(p)$, for $i = 0, ..., [\pi/\ell]$, where $[t] := \max\{k \in \mathbb{Z} : k \le t\}$. Since the subbundles $\mathcal{N}^-$ and $\mathcal{N}^+$ are 1-dimensional, we have that

$$\begin{aligned}
\frac{\left\|\Phi^{\pi}_{H_2}(p)|_{\mathcal{N}^-_p}\right\|}{\left\|\Phi^{\pi}_{H_2}(p)|_{\mathcal{N}^+_p}\right\|} &= \frac{\left\|\Phi^{\pi-\ell[\pi/\ell]+\ell[\pi/\ell]}_{H_2}(p)|_{\mathcal{N}^-_p}\right\|}{\left\|\Phi^{\pi-\ell[\pi/\ell]+\ell[\pi/\ell]}_{H_2}(p)|_{\mathcal{N}^+_p}\right\|} \\
&= \frac{\left\|\Phi^{\pi-\ell[\pi/\ell]}_{H_2}(p)|_{\mathcal{N}^-_p}\right\|}{\left\|\Phi^{\pi-\ell[\pi/\ell]}_{H_2}(p)|_{\mathcal{N}^+_p}\right\|} \cdot \prod_{i=1}^{[\pi/\ell]} \frac{\left\|\Phi^{\ell}_{H_2}(p_i)|_{\mathcal{N}^-_{p_i}}\right\|}{\left\|\Phi^{\ell}_{H_2}(p_i)|_{\mathcal{N}^+_{p_i}}\right\|} \\
&\le C(p, H_2) \cdot \left(\frac{1}{2}\right)^{[\pi/\ell]},
\end{aligned} \tag{3.2}$$

where

$$C(p, H_2) := \sup\left\{0 \le t \le \ell : \left\|\Phi^{t}_{H_2}(p)|_{\mathcal{N}^-_p}\right\| \cdot \left\|\Phi^{t}_{H_2}(p)|_{\mathcal{N}^+_p}\right\|^{-1}\right\}$$

depends continuously on H_2 in the C^2 topology. Then, there exists a uniform bound for $C(p, \cdot)$, for any Hamiltonian that is C^2-close to H.

If, in Lemma 3.5, we let r be small enough, we can choose $\pi > T$ arbitrarily large. So, inequality (3.2) ensures that

$$\frac{1}{\pi}\log\left\|\Phi^{\pi}_{H_2}(p)|_{\mathcal{N}^-_p}\right\| \le \frac{1}{\pi}\log C(p, H_2) + \frac{[\pi/\ell]}{\pi}\log\frac{1}{2} + \frac{1}{\pi}\log\left\|\Phi^{\pi}_{H_2}(p)|_{\mathcal{N}^+_p}\right\|.$$

Moreover, since $\left\|\Phi^{\pi}_{H_2}(p)\right\| = \left\|\Phi^{\pi}_{H_2}(p)|_{\mathcal{N}^+_x}\right\|$ and $\Phi^{\pi}_{H_2}$ is conservative, the sum of the Lyapunov exponents is equal to zero, that is,

$$\frac{1}{\pi}\log\left\|\Phi^{\pi}_{H_2}(p)|_{\mathcal{N}^-_p}\right\| = -\frac{1}{\pi}\log\left\|\Phi^{\pi}_{H_2}(p)|_{\mathcal{N}^+_p}\right\|.$$

Thus,

$$\frac{2}{\pi}\log\left\|\Phi^{\pi}_{H_2}(p)\right\| = \frac{2}{\pi}\log\left\|\Phi^{\pi}_{H_2}(p)|_{\mathcal{N}^+_p}\right\| \geq -\frac{1}{\pi}\log C(p,H_2) - \frac{[\pi/\ell]}{\pi}\log\frac{1}{2}$$
$$\geq -\frac{1}{\pi}\log C(p,H_2) + \frac{1}{\ell}\log 2.$$

Notice that the constants involved in the inequality (3.2) do not depend on π. Then, we can choose the period of p large enough such that

$$\frac{1}{\pi}\log\left\|\Phi^{\pi}_{H_2}(p)\right\| \geq \frac{1}{2\ell}\log 2 > \delta.$$

This contradicts expression (3.1). Thus, Φ^t_H admits a dominated splitting over $\mathcal{E}^{\star}_{H,e}$. □

Remark 13 *It follows from the previous proof that the conclusion of Lemma 3.9 also holds if we assume that the energy surface $\mathcal{E}_{H,e}$ is regular and far from elliptic orbits, and the same holds for any analytic continuation of it, instead of belonging to $\mathcal{G}^2_\mu(M^4)$.*

Lemma 3.10 *Consider a Hamiltonian system $(H,e,\mathcal{E}_{H,e})$, where $\mathcal{E}_{H,e}$ is a 3-dimensional energy surface. If Φ^t_H admits a dominated splitting over $\mathcal{E}_{H,e}$ then $(H,e,\mathcal{E}_{H,e})$ is an Anosov Hamiltonian system.*

Proof: Consider a Hamiltonian system $(H,e,\mathcal{E}_{H,e})$ such that Φ^t_H admits a dominated splitting $\mathcal{N}_x = \mathcal{N}^-_x \oplus \mathcal{N}^+_x$, for any $x \in \mathcal{E}_{H,e}$. By Definition 3.5, we have that there exist $\ell \in \mathbb{N}$ and a constant $\theta \in (0,1)$ such that

$$\Delta(x,\ell) := \|\Phi^{\ell}_H(x)|_{\mathcal{N}^-_x}\|\,\|\Phi^{-\ell}_H\big(X^{\ell}_H(x)\big)|_{\mathcal{N}^+_{X^{\ell}_H(x)}}\| \leq \theta, \quad \forall\, x \in \mathcal{E}_{H,e}.$$

Observe that, by the chain rule, we have $\Delta(x,i\ell) \leq \theta^i$, for any $i \in \mathbb{N}$. Furthermore, every $t>0$ can be written as $t = i\ell + r$, where $r \in [0,\ell)$. Since M is compact, we have that $\|\Phi^r_H\|$ is bounded, say by L. So, defining $C := \theta^{-\frac{r}{\ell}}L^2$ and $\kappa := \theta^{\frac{1}{\ell}}$, we want to prove that C and κ are directly related with the constants associated to the hyperbolicity of $\mathcal{E}_{H,e}$. In fact, for every $x \in \mathcal{E}_{H,e}$ and $t>0$, we have that

$$\begin{aligned}
\Delta(x,t) &= \|\Phi^{i\ell+r}_H(x)|_{\mathcal{N}^-_x}\|\|\Phi^{-i\ell-r}_H\big(X^{i\ell+r}_H(x)\big)|_{\mathcal{N}^+_{X^{i\ell+r}_H(x)}}\| \\
&= \|\Phi^{i\ell}_H\big(X^r_H(x)\big)|_{\mathcal{N}^-_{X^r_H(x)}}\|\|\Phi^r_H(x)|_{\mathcal{N}^-_x}\| \cdot \\
&\quad \cdot \|\Phi^{-i\ell}_H\big(X^{i\ell}_H(x)\big)|_{\mathcal{N}^+_{X^{i\ell}_H(x)}}\|\|\Phi^{-r}_H\big(X^{i\ell+r}_H(x)\big)|_{\mathcal{N}^+_{X^{i\ell+r}_H(x)}}\| \\
&\leq L^2\,\Delta(x,i\ell) \leq L^2\,\theta^i = L^2\,\theta^{\frac{t-r}{\ell}} = \theta^{\frac{-r}{\ell}}L^2\,\theta^{\frac{t}{\ell}} = C\,\kappa^t.
\end{aligned}$$

Denote by α_t the angle associated to the fibers $\mathcal{N}^-_{X^t_H(x)}$ and $\mathcal{N}^+_{X^t_H(x)}$ and notice that, by domination, there exists $\beta > 0$ such that $\alpha_t \geq \beta$, for any t. Since $\mathcal{E}_{H,e}$ is regular and compact, there is $K > 1$ such that, for every $x \in \mathcal{E}_{H,e}$, $K^{-1} \leq \|X_H(x)\| \leq K$. Since Φ^t_H is conservative and the subbundles $\mathcal{N}^-$ and $\mathcal{N}^+$ are both 1-dimensional, we have that

$$\sin(\alpha_0) = \left\|\Phi^t_H(x)|_{\mathcal{N}^-_x}\right\| \left\|\Phi^t_H(x)|_{\mathcal{N}^+_x}\right\| \sin(\alpha_t) \frac{\|X_H(X^t_H(x))\|}{\|X_H(x)\|}.$$

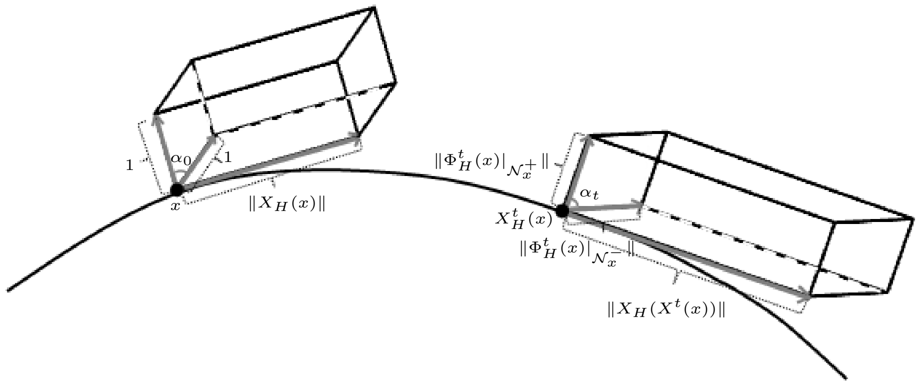

Figure 3.12: Preservation of the volume of a box.

Given $t > 0$, as $0 < \beta \leq \alpha_t < \dfrac{\pi}{2}$, it follows that $\sin(\alpha_t) \geq \sin(\beta)$. So, taking a positive $C_1 := \sin(\beta)^{-1} K^2 C$, for any $x \in \mathcal{E}_{H,e}$ and any $t > 0$, we have that

$$\begin{aligned}\left\|\Phi^t_H(x)|_{\mathcal{N}^-_x}\right\|^2 &= \frac{\sin(\alpha_0)}{\sin(\alpha_t)} \, \frac{\|X_H(x)\|}{\|X_H(X^t_H(x))\|} \left\|\Phi^{-t}_H(x)|_{\mathcal{N}^+_x}\right\| \left\|\Phi^t_H(x)|_{\mathcal{N}^-_x}\right\| \\ &\leq \sin(\beta)^{-1} K^2 \, \Delta(x,t) \leq \sin(\beta)^{-1} K^2 \, C \, \kappa^t \\ &= C_1 \, \kappa^t.\end{aligned}$$

Analogously, for any $x \in \mathcal{E}_{H,e}$ and for any $t > 0$, it follows that

$$\begin{aligned}\left\|\Phi^{-t}_H(x)|_{\mathcal{N}^+_x}\right\|^2 &= \frac{\sin(\alpha_t)}{\sin(\alpha_0)} \, \frac{\|X_H(X^t_H(x))\|}{\|X_H(x)\|} \, \Delta(x,t) \\ &\leq \sin(\beta)^{-1} K^2 \, C \, \kappa^t \\ &= C_1 \, \kappa^t.\end{aligned}$$

These two inequalities show that $\mathcal{E}_{H,e}$ is uniformly hyperbolic for the transversal linear Poincaré flow. Then, by Lemma 3.1, $\mathcal{E}_{H,e}$ is uniformly hyperbolic for X^t_H, meaning that $(H, e, \mathcal{E}_{H,e}) \in \mathcal{A}^2_\omega(M^4)$. $\square$

Remark 14 *Recall that, if the energy e is regular then $H^{-1}(\{e\})$ decomposes into a finite number of regular energy hypersurfaces. If each one of these energy hypersurfaces belongs to $\mathcal{G}^2_\omega(M^4)$ then, by Theorem 4, we can prove that the energy level set $H^{-1}(\{e\})$ is Anosov.*

3.3.3 Structural stability conjecture

In this section, we prove that C^2-structurally stable 4-dimensional Hamiltonian systems are Anosov.

Theorem 6 ([13, Theorem 2]) If $(H, e, \mathcal{E}_{H,e})$ is a structurally stable Hamiltonian system then $(H, e, \mathcal{E}_{H,e}) \in \mathcal{A}^2_\omega(M^4)$.

Proof: Let $(H, e, \mathcal{E}_{H,e})$ be a C^2-structurally stable Hamiltonian system. Then, there exist a C^2-neighborhood $\mathcal{U}$ of H and $\epsilon > 0$ such that, for any $\tilde{e} \in (e-\epsilon, e+\epsilon)$ and any $\tilde{H} \in \mathcal{U}$, the analytic continuation $\mathcal{E}_{\tilde{H},\tilde{e}}$ is well-defined and there exists a homeomorphism $h\colon \mathcal{E}_{H,e} \to \mathcal{E}_{\tilde{H},\tilde{e}}$ preserving orbits and their orientation. In particular, since $\mathcal{E}_{H,e}$ is regular, $\mathcal{E}_{\tilde{H},\tilde{e}}$ is also regular. By contradiction, suppose that $(H, e, \mathcal{E}_{H,e})$ is not an Anosov Hamiltonian system. Therefore, by Lemma 3.10, $\mathcal{E}_{H,e}$ does not admit a dominated splitting. Hence, as explained in Remark 13, there exist $\tilde{H} \in \mathcal{U}$ and $\tilde{\epsilon} \in (e-\epsilon, e+\epsilon)$ such that the analytic continuation $\mathcal{E}_{\tilde{H},\tilde{e}}$ of $\mathcal{E}_{H,e}$ has an elliptic closed orbit. Moreover, it follows from the proof of Lemma 3.9 that this orbit can be chosen with arbitrarily large period. Now, applying Frank's Lemma for Hamiltonians (see [78]) several times, the idea is to concatenate small rotations, in order to get $\bar{H} \in \mathcal{U}$ and $\bar{\epsilon} \in (e-\epsilon, e+\epsilon)$ such that the analytic continuation $\mathcal{E}_{\bar{H},\bar{e}}$ of $\mathcal{E}_{\tilde{H},\tilde{e}}$ exhibits a parabolic closed orbit. We formalize now this argument.

Consider an elliptic closed orbit p of $\tilde{H}$, with arbitrarily large period $\tilde{\pi} \in \mathbb{N}$, and $\theta \in [0, \widetilde{\pi/2}]$ such that $\rho = \exp(\theta i)$ is a eigenvalue of $\Phi^{\tilde{\pi}}_{\tilde{H}}(p)$. Fix $\epsilon > 0$ and $\tau > 0$ and let $\delta > 0$ be the constant given by Frank's Lemma for Hamiltonians (Lemma 3.7). We write the period $\tilde{\pi}$ as $\tilde{\pi} = \dfrac{\theta}{\alpha}$, where $0 < \alpha < \delta$.

Recall that the special linear group $SL(2,\mathbb{R})$ is the group of all real 2×2 matrices with determinant of modulus equal to 1 and notice that, once we are in the two-dimensional case, the symplectic setting is nothing more than the conservative one. Therefore, let

R_α be the rotation matrix of angle α, where α is chosen such that R_α is C^0-close to the identity. We observe that $\Phi^{\tilde{\pi}}_{\tilde{H}}(p)$ can be seen as R_θ. So, by Frank's Lemma for Hamiltonians (Lemma 3.7), for $i = 1, ..., \tilde{\pi}$, for any flowbox V_i of an injective arc of orbit $X^{[i-1,i]}_{\tilde{H}}(p)$ and for a transversal symplectic δ-perturbation F_i of $\Phi^1_{\tilde{H}}(X^{i-1}_{\tilde{H}}(p))$, there exists $H_i \in C^2(M, \mathbb{R})$ satisfying:

- $H_i \in \mathcal{U}$ is C^2-close to $\tilde{H}$;
- $\Phi^1_{H_i}(X^{i-1}_{\tilde{H}}(p)) = F_i$;
- $\tilde{H} = H_i$ on $X^{[0,1]}_{\tilde{H}}(X^i_{\tilde{H}}(p)) \cup (M \backslash V_i)$.

Let

$$F_i := \Phi^i_{\tilde{H}}(p) \circ R_{-\alpha} \circ \left[\Phi^{i-1}_{\tilde{H}}(p)\right]^{-1}$$

and note that F_i is a symplectomorphism, since $det\ F_i = 1$. We define $\bar{H} = \tilde{H}$, on $M \setminus \bigcup_{i=1}^{\tilde{\pi}} V_i$, and $\bar{H} = H_i$, on V_i, for $i \in \{1, ..., \tilde{\pi}\}$. Now, observe that

$$\begin{aligned} \Phi^{\tilde{\pi}}_{\bar{H}}(p) &= F_{\tilde{\pi}} \circ F_{\tilde{\pi}-1} \circ \cdots \circ F_2 \circ F_1 = \Phi^{\tilde{\pi}}_{\tilde{H}}(p) \circ R_{-\tilde{\pi}\alpha} \\ &= \Phi^{\tilde{\pi}}_{\tilde{H}}(p) \circ R_{-\theta} = id. \end{aligned}$$

Thus, assuming that $(H, e, \mathcal{E}_{H,e})$ is a C^2-structurally stable Hamiltonian system, but not an Anosov Hamiltonian system, we constructed $\bar{H} \in \mathcal{U}$ with a parabolic closed orbit p. But this is a contradiction, since the presence of a parabolic closed orbit prevents the structural stability (see [70]). Then, a 4-dimensional C^2-structurally stable Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is Anosov. $\square$

Remark 15 *Notice that Robinson, whilst using different techniques, also proved that the existence of an elliptic periodic point prevents the structural stability (see [70, Theorem 6.4]).*

3.3.4 Boundary of $\mathcal{A}^2_\omega(M^4)$

As a consequence of Theorem 4, we prove that the boundary of $\mathcal{A}^2_\omega(M^4)$ has no isolated points, as stated in Corollary 2.

By contradiction, let $(H, e, \mathcal{E}_{H,e})$ be an isolated point on the boundary of $\mathcal{A}^2_\omega(M^4)$. This means that $\mathcal{E}_{H,e}$ is not uniformly hyperbolic, but any analytic continuation $\mathcal{E}_{\tilde{H},\tilde{e}}$ is uniformly hyperbolic, for any $\tilde{H}$ arbitrarily close to H and for any $\tilde{e}$ in a small neighborhood of e. In this case, we claim:

Claim 3.2 *If $(H, e, \mathcal{E}_{H,e})$ is an isolated point on the boundary of $\mathcal{A}^2_\omega(M^4)$ then $\mathcal{E}_{H,e}$ has no singularities.*

If this claim is not true, we can find a singularity q in the energy surface $\mathcal{E}_{H,e}$, which can be hyperbolic, or not. If q is hyperbolic then, since $(H, e, \mathcal{E}_{H,e})$ is isolated on the boundary of $\mathcal{A}^2_\omega(M^4)$, an adequate perturbation of $(H, e, \mathcal{E}_{H,e})$ will derive a Hamiltonian system $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}})$ in $\mathcal{A}^2_\omega(M^4)$ with a singularity in $\mathcal{E}_{\tilde{H},\tilde{e}}$, since hyperbolic singularities persist to small perturbations. But this is a contradiction. Now, assuming that the singularity q is not hyperbolic, by a small adequate perturbation of $(H, e, \mathcal{E}_{H,e})$, we can make it hyperbolic, which, as before, is a contradiction. So, the claim is true.

Now, to conclude the proof of Corollary 2, we follow the ideas presented in the proof of Theorem 4. Observe that, by Claim 3.2, the energy surface $\mathcal{E}_{H,e}$ is regular. So, we start by proving that Φ^t_H admits a dominated splitting over $\mathcal{E}_{H,e}$. Recall that, in Lemma 3.9, the main step is achieved because, given that $(H, e, \mathcal{E}_{H,e}) \in \mathcal{G}^2_\omega(M^4)$, elliptic orbits are not allowed in $\mathcal{E}_{H,e}$, and the same holds for the analytic continuations of $\mathcal{E}_{H,e}$. However, even without this assumption, we can go on with an identical argument. In fact, since $(H, e, \mathcal{E}_{H,e})$ is an isolated point on the boundary of $\mathcal{A}^2_\omega(M^4)$, any perturbed Hamiltonian system $(\tilde{H}, \tilde{e}, \mathcal{E}_{\tilde{H},\tilde{e}})$, arbitrarily close to $(H, e, \mathcal{E}_{H,e})$, will be in $\mathcal{A}^2_\omega(M^4)$, preventing the existence of elliptic closed orbits on $\mathcal{E}_{\tilde{H},\tilde{e}}$. Therefore, we also conclude that Φ^t_H admits a dominated splitting over $\mathcal{E}_{H,e}$. Now, by Lemma 3.10, we have that the Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is Anosov. But this is a contradiction, because we took $(H, e, \mathcal{E}_{H,e})$ on the boundary of the open set $\mathcal{A}^2_\omega(M^4)$. So, the boundary of $\mathcal{A}^2_\omega(M^4)$ cannot have isolated points. $\square$

3.3.5 Auxiliary lemmas

In this section, we state the proof of some auxiliary results for Hamiltonian systems defined on a $2d$-dimensional symplectic manifold, for $d \geq 2$. The first one (Lemma 3.11) asserts that, C^2-generically, the quotient between the period of two distinct closed orbits of a Hamiltonian is irrational. After, in Lemma 3.12, we show that, given a C^2-generic

Hamiltonian H, there exists an open and dense set in $H(M)$ such that every energy taken in such a set is regular. Afterwards, we show that, given a C^2-generic Hamiltonian H, there exists an open and dense set in $H(M)$ such that, for every energy e taken in such a set, the Hamiltonian level (H, e) is transitive (Lemma 3.13).

Lemma 3.11 *There is a residual $\mathcal{R}$ in $C^2(M, \mathbb{R})$ such that, for any $H \in \mathcal{R}$, any distinct $p, q \in Per(H)$, with periods π_p and π_q, satisfy $\dfrac{\pi_p}{\pi_q} \in \mathbb{R} \backslash \mathbb{Q}$.*

Proof: Fix $n \in \mathbb{N}$. By Theorem 3.5, the following set

$$\mathcal{A}_n := \big\{H \in C^2(M, \mathbb{R}) : Sing(H) \text{ and } Per_n(H) \text{ do not satisfy non-trivial resonances}\big\}$$

is open and dense in $C^2(M, \mathbb{R})$. Also, define the open set

$$\mathcal{B}_n := \left\{H \in \mathcal{A}_n : \text{ if } p, q \in Per_n(H) \text{ and } p \neq q \text{ then } \frac{\pi_p}{\pi_q} \notin \{r_i\}_{i=1}^n\right\},$$

where $\{r_i\}_{i=1}^\infty$ denote the positive rational numbers, with a fixed order.

Now, this proof follows the ideas stated in the proof of [11, Lemma 2.2], but using the version of the Pasting Lemma for Hamiltonians, proved in Lemma 2.3.

Fix $\epsilon > 0$ and $H_1 \in C^2(M, \mathbb{R})$. By density of $\mathcal{A}_n$, there is $H_2 \in \mathcal{A}_n$, ϵ-C^2-close to H_1. Recall that, by Proposition 3.2, the closed orbits with period less or equal than n of H_2 are uniformly avoidable, and so isolated. So, $Per_n(H_2)$ has a finite number of elements, say $\{p_i\}_{i=1}^m$, for fixed $m \in \mathbb{N}$.

Given a positive sequence $\{s_i\}_{i=1}^m$, the vector field $X_{\overline{H}_i} = \frac{1}{s_i+1} X_{H_2}$ is also a divergence-free vector field, for any $1 \leq i \leq m$. Observe that if we choose s_i arbitrarily close to 0 then $X_{\overline{H}_i}$ is ϵ-C^2-close to X_{H_2}.

For any $1 \leq i \leq m$, consider tubular compact neighborhoods K_i of p_i, sufficiently small such that some open neighborhoods $\mathcal{W}_i$ of K_i are pairwise disjoint. The idea now is to apply, recursively, the Pasting Lemma for Hamiltonians (Lemma 3.6), in order to define $\tilde{H}_m \in C^2(M, \mathbb{R})$ such that:

- $\tilde{H}_m$ is ϵ-C^2-close to H_2, as s_i converges to 0;
- $\pi_{\tilde{H}_m, p_i} = (1 + s_i)\pi_{H_2, p_i}$, for $1 \leq i \leq m$.

By a good small choice of the sequence $\{s_i\}_{i=1}^m$, we have that $\tilde{H}_m \in \mathcal{A}_n$ and that $\frac{\pi_{\tilde{H}_m,p_i}}{\pi_{\tilde{H}_m,p_j}} \notin \{r_i\}_{i=1}^n$, for $i \neq j$. Thus, $\tilde{H}_m \in \mathcal{B}_n$.

Since $\mathcal{B}_n$ is open and dense in $C^2(M,\mathbb{R})$, for any $n \in \mathbb{N}$, the desired residual subset of $C^2(M,\mathbb{R})$ is given by $\mathcal{R} := \cap_{n\in\mathbb{N}}\mathcal{B}_n$. $\square$

Lemma 3.12 *There is an open and dense set $\mathcal{O}$ in $C^2(M,\mathbb{R})$ such that, for any $H \in \mathcal{O}$, there is an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that, for any energy $e \in \mathcal{S}(H)$, the Hamiltonian level (H,e) is regular.*

Proof: First, let us observe that Morse functions are C^2-open and dense in $C^2(M,\mathbb{R})$ and that a Morse function, defined on a compact manifold, admits only finitely many critical points (see, for instance, [54]). Let $\mathcal{O}$ be the open and dense set of Morse functions in $C^2(M,\mathbb{R})$. So, any $H \in \mathcal{O}$ has a finite number of singularities and, therefore, $H(M)$ has a finite number of non-regular elements. Fix $H \in \mathcal{R}$ and define the C^1-open set,

$$\mathcal{S}(H) := \{e \in H(M) : X_H(p) \neq 0, \text{ for any } p \in H^{-1}(\{e\})\}.$$

We just have to prove that $\mathcal{S}(H)$ is dense in $H(M)$, that is, for any $\delta > 0$ and any $e \in H(M)$, there is $\tilde{e} \in \mathcal{S}(H)$ such that $|e - \tilde{e}| < \delta$. So, fix $\delta > 0$ and $e \in H(M)$. Note that if there exists $p \in H^{-1}(\{e\})$ such that $X(p) = 0$ then it is enough to δ-perturb e to $\tilde{e}$, in order to have $X_H(p) \neq 0$, for any $p \in H^{-1}(\{\tilde{e}\})$. Thus, $\tilde{e} \in \mathcal{S}(H)$. $\square$

Lemma 3.13 *There is a residual set $\mathcal{R}$ in $C^2(M,\mathbb{R})$ such that, for any $H \in \mathcal{R}$, there is an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that, for every $e \in \mathcal{S}(H)$,*

- *$H^{-1}(\{e\})$ is regular;*
- *the closed orbits of H in $H^{-1}(\{e\})$ do not satisfy non-trivial resonances;*
- *the Hamiltonian level (H,e) is transitive.*

Proof: Let $\mathcal{R}_0$ be the residual set given by Theorem 3.5 and consider $\mathcal{O}$ and $\mathcal{S}(H)$, for $H \in \mathcal{O}$, as in Lemma 3.12. Observe that, if $e \in \mathcal{S}(H)$ then $H^{-1}(\{e\}) = \sqcup_{i=1}^{I_e}\mathcal{E}_{H,e,i}$. In this case, let $\{U_n\}_n$ be a countable basis of open sets on M. Fix $1 \leq i \leq I_e$ and define $U_n^i := U_n \cap \mathcal{E}_{H,e,i}$, whenever non-empty. So, $\{U_n^i\}_n$ is a countable basis of open sets on $\mathcal{E}_{H,e,i}$. We say that $H \in \mathcal{P}_{n,m,i,e}$ if

$$\left[\cup_{t>0} X_H^t(U_n^i)\right] \cap U_m^i \neq \emptyset.$$

Now, we define the residual set

$$\mathcal{R} := \mathcal{R}_0 \cap \mathcal{O} \cap \bigcap_{n,m} \big(\mathcal{P}_{n,m,i,e} \cup (\overline{\mathcal{P}_{n,m,i,e}})^c\big),$$

where, given a set S, $\bar{S}$ stands for its closure and S^c for its complementary.

Fix $H \in \mathcal{R}$, $e \in \mathcal{S}(H)$ and $1 \leq i \leq I_e$. Thus, $H^{-1}(\{e\})$ is regular and any closed orbit of H in $\mathcal{E}_{H,e,i}$ do not satisfy non-trivial resonances. Moreover, for all integers n and m, we have that $H \in \mathcal{P}_{n,m,i,e}$ or $H \in (\overline{\mathcal{P}_{n,m,i,e}})^c$. Observe that if $H \in \mathcal{P}_{n,m,i,e}$, for all integers n and m and any $1 \leq i \leq I_e$, then (H, e) is transitive.

So, by contradiction, assume that there are some integers n and m and $1 \leq i \leq I_e$ such that $H \in \overline{\mathcal{P}_{n,m,i,e}}^{\,c}$. Choose $x \in U_n^i$ and $y \in U_m^i$. By Remark 10, all points $x, y \in \mathcal{E}_{H,e,i}$ are connected by an ϵ-pseudo-orbit, for any $\epsilon > 0$. Moreover, since $H \in \mathcal{R}_0$, we can apply the Connecting Lemma for pseudo-orbits of Hamiltonians (Lemma 1). So, for any C^2-neighborhood $\mathcal{U}$ of H, there exists $\tilde{H} \in \mathcal{U} \cap \mathcal{R}_0 \cap \mathcal{O} \cap \overline{\mathcal{P}_{n,m,\tilde{i},e}}^{\,c}$ such that $e = \tilde{H}(x)$, where $U_n^{\tilde{i}}$ and $U_m^{\tilde{i}}$ are elements of the basis of the well-defined analytic continuation $\mathcal{E}_{\tilde{H},e,\tilde{i}}$ of $\mathcal{E}_{H,e,i}$ such that $x \in U_n^{\tilde{i}}$ and $y \in U_m^{\tilde{i}}$, and there is $T > 0$ such that $X_{\tilde{H}}^T(x) = y$ on $\mathcal{E}_{\tilde{H},e,\tilde{i}}$. Then $\tilde{H} \in \mathcal{P}_{m,n,\tilde{i},e}$, which is a contradiction. Hence $H \in \mathcal{P}_{n,m,i,e}$, for all integers n and m and for any $1 \leq i \leq I_e$. Therefore, (H, e) is transitive, for any $H \in \mathcal{R}$ and any $e \in \mathcal{S}(H)$. $\square$

3.3.6 Energy hypersurfaces as homoclinic classes

In this section, we want to prove the following corollary of Lemma 3.13.

Corollary 4 There is a residual set $\mathcal{R}$ in $C^2(M, \mathbb{R})$ such that, for any $H \in \mathcal{R}$, there is an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that if $e \in \mathcal{S}(H)$ then any energy hypersurface of $H^{-1}(\{e\})$ is a homoclinic class.

Proof: Let $\mathcal{R}$ and $\mathcal{S}(H)$, for $H \in \mathcal{R}$, be as in Lemma 3.13. Recall that if $e \in \mathcal{S}(H)$ then $H^{-1}(\{e\}) = \sqcup_{i=1}^{I_e} \mathcal{E}_{H,e,i}$ and, fixing $1 \leq i \leq I_e$, we can define a countable basis of open sets $\{U_n^i\}_n$ on the energy hypersurface $\mathcal{E}_{H,e,i}$.

Let $\tilde{\mathcal{R}}$ denote the C^2-residual set in $C^2(M,\mathbb{R})$ such that, for any $H \in \tilde{\mathcal{R}}$, $Per(H)$ are hyperbolic.

Fix $H \in \mathcal{R} \cap \tilde{\mathcal{R}}$ and take a C^2-neighborhood $\mathcal{U}$ of H such that the analytic continuation $p_{\tilde{H}}$ of a hyperbolic closed orbit p_H of H is well-defined, for any $\tilde{H} \in \mathcal{U}$. So, for any integer n, define the open sets

$$\mathcal{W}_n := \{\tilde{H} \in \mathcal{U} : W^{s,u}_{\tilde{H}}(p_{\tilde{H}}) \cap U^i_n \neq \emptyset\}.$$

We want to show that $\mathcal{W}_n$ is a dense subset of $\mathcal{U}$, for any $n \in \mathbb{N}$. First, observe that $\mathcal{R}_{\mathcal{U}} := \mathcal{R} \cap \mathcal{U}$ is a dense subset of $\mathcal{U}$ such that, for any $\tilde{H} \in \mathcal{R}_{\mathcal{U}}$, there is an open and dense set $\mathcal{S}(\tilde{H}) \subset \tilde{H}(M)$ such that any $e \in \mathcal{S}(\tilde{H})$ is regular and $(tildeH, e)$ is transitive. So, fixing $n \in \mathbb{N}$, for any $\tilde{H} \in \mathcal{R}_{\mathcal{U}}$ and any neighborhood V of a hyperbolic closed orbit $p_{\tilde{H}}$ there exist $j, k > 0$ satisfying $X^j_{\tilde{H}}(V) \cap U^{\tilde{i}}_n \neq \emptyset$ and $X^{-k}_{\tilde{H}}(V) \cap U^{\tilde{i}}_n \neq \emptyset$, where $\{U^{\tilde{i}}_n\}_n$ is a countable basis of open sets on $\mathcal{E}_{\tilde{H},e,\tilde{i}}$. By Hayashi's Connecting Lemma of Hamiltonians (see [79]), there exists a Hamiltonian $\bar{H}$, C^2-close to $\tilde{H}$, such that $\bar{H} \in \mathcal{W}_n$. Hence, $\mathcal{W}_n$ is dense on $\mathcal{U}$, for any $n \in \mathbb{N}$. Therefore,

$$\mathcal{W} := \bigcap_{n\in\mathbb{N}} \mathcal{W}_n = \left\{\tilde{H} \in \mathcal{U} : \overline{W^{s,u}_{\tilde{H}}(p_{\tilde{H}})} = \mathcal{E}_{\tilde{H},e,\tilde{i}}\right\}$$

is a residual subset of $\mathcal{U}$.

Fix $\tilde{H} \in \mathcal{R} \cap \mathcal{W}$ and $e \in \mathcal{S}(\tilde{H})$. Let $\{U^{\tilde{i}}_n\}_n$ be a countable basis of open sets on the energy hypersurface $\mathcal{E}_{\tilde{H},e,\tilde{i}}$ of $\tilde{H}^{-1}(\{e\})$. Fix $n \in \mathbb{N}$ and a hyperbolic closed orbit $p_{\tilde{H}}$ of $\tilde{H}$. Observe that any non-periodic $x \in U^{\tilde{i}}_n$ is an accumulation point of $W^{s,u}_{\tilde{H}}(p_{\tilde{H}})$. By the Connecting Lemma for Hamiltonians (see [79]), we construct homoclinic intersections on $U^{\tilde{i}}_n$ and, by a small C^2-perturbation, we turn it transversal. So, the set

$$\mathcal{Z}_n := \{\tilde{H} \in \mathcal{U} : p_{\tilde{H}} \text{ has a homoclinic transversal intersection on } U^{\tilde{i}}_n\}$$

is open and dense on $\mathcal{U}$, for any $n \in \mathbb{N}$. Therefore, the set

$$\mathcal{Z} := \bigcap_{n\in\mathbb{N}} \mathcal{Z}_n = \{\tilde{H} \in \mathcal{U} : \mathcal{H}_{p_{\tilde{H}},\tilde{H}} = \mathcal{E}_{\tilde{H},e,\tilde{i}}\}$$

is residual in $\mathcal{U}$. Observe that this is valid for any small C^2-neighborhood $\mathcal{U}$ of H in $\mathcal{R} \cap \tilde{\mathcal{R}}$. So, the set

$$\mathcal{R}_1 := \{H \in C^2(M,\mathbb{R}) \cap \mathcal{R} : \mathcal{H}_{p_H,H} = \mathcal{E}_{H,e,i}\}$$

is residual in $C^2(M,\mathbb{R})$, for any $1 \leq i \leq I_e$. Thus, there is a residual set $\mathcal{R}_1$ in $C^2(M,\mathbb{R})$ such that, for any $H \in \mathcal{R}_1$, there is an open and dense set $\mathcal{S}(H)$ such that, for $e \in \mathcal{S}(H)$, any energy hypersurface of $H^{-1}(\{e\})$ is a homoclinic class. $\square$

3.3.7 Generic topological mixing

In this section, we conclude the proof of Theorem 10.

Theorem 10 There is a residual $\mathcal{R}$ in $C^2(M,\mathbb{R})$ such that, for any $H \in \mathcal{R}$, there is an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that, for every $e \in \mathcal{S}(H)$, the Hamiltonian level (H,e) is topologically mixing.

Proof: Let $\mathcal{R}_0$ be the residual set given by Lemma 3.11, $\mathcal{R}_1$ be the residual set given by Lemma 3.13 and $\mathcal{R}_2$ be the residual set given by Corollary 4. Define

$$\mathcal{R} := \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}_2.$$

Now, we follow the ideas on the proof of [1, Theorem B], making the necessary adaptations to the Hamiltonian setting.

Fix $H \in \mathcal{R}$. Since $H \in \mathcal{R}_1$, by Lemma 3.13, there is an open and dense set $\mathcal{S}(H)$ such that, for any $e \in \mathcal{S}(H)$, the Hamiltonian level (H,e) is transitive. So, to conclude the proof of Theorem 10, we just have to prove that, for any $e \in \mathcal{S}(H)$, the Hamiltonian level (H,e) is topologically mixing.

Fix $e \in \mathcal{S}(H)$ and let $\mathcal{E}_{H,e,i}$ be an energy hypersurface of $H^{-1}(\{e\})$, for $1 \leq i \leq I_e$. Let us prove that $\mathcal{E}_{H,e,i}$ is topologically mixing, that is, for any open, nonempty subsets U and V of $\mathcal{E}_{H,e,i}$, there is $\tau \in \mathbb{R}$ such that $X_H^t(U) \cap V \neq \emptyset$, for any $t \geq \tau$.

Given that $H \in \mathcal{R}_2$ and $e \in \mathcal{S}(H)$, by Corollary 4, $\mathcal{E}_{H,e,i}$ is a homoclinic class. Since hyperbolic closed orbits with the same index are dense in the homoclinic class, we can find two different hyperbolic closed orbits γ_1 and γ_2 of H, with period π_p and π_q, where $p \in \gamma_1$ and $q \in \gamma_2$, such that $ind(\gamma_1) = ind(\gamma_2)$ and $\gamma_1 \cap U \neq \emptyset$ and $\gamma_2 \cap V \neq \emptyset$. Moreover, since $H \in \mathcal{R}_0$, we have that $\frac{\pi_p}{\pi_q} \in \mathbb{R} \backslash \mathbb{Q}$.

Fix $x \in \gamma_1 \cap U$, $y \in \gamma_2 \cap V$ and $z \in W^u(x) \cap W^s(y)$. Thus, there is $\tau_1 > 0$ such that

- $\{X_H^{-(\tau_1 + m\pi_p)}(z)\}_{m \in \mathbb{N}} \subset W^u(x)$;

- $\lim_{m\to+\infty} X_H^{-(\tau_1+m\pi_p)}(z) = x$.

Then, there is $t_1 > 0$ such that $X_H^{-(t_1+m\pi_p)}(z) \in U$ and, therefore, $z \in X_H^{t_1+m\pi_p}(U)$, for every $m \in \mathbb{N}$. Similarly, there is $t_2 > 0$ and a small $\epsilon > 0$ such that $X_H^{t_2+n\pi_q+s}(z) \in V$, for every $n \in \mathbb{N}$ and $|s| < \epsilon$.

Since $\frac{\pi_p}{\pi_q} \in \mathbb{R}\backslash\mathbb{Q}$, observe that the set $\{m\pi_p + n\pi_q + s : m, n \in \mathbb{Z}, |s| < \epsilon\}$ contains an interval of the form $[T, +\infty)$, for some $T > 0$. This follows from the transitivity of the future orbits of irrational rotations of the circle. Hence, for any $t \geq t_1 + t_2 + T$, there are $m, n \in \mathbb{N}$ and $|s| < \epsilon$ such that $t = t_1 + t_2 + m\pi_p + n\pi_q + s$. Then, $X_H^{t_2+n\pi_q+s}(z) \in X_H^t(U) \cap V$, for any $t \geq t_1 + t_2 + T$. So, $\mathcal{E}_{H,e,i}$ is a topologically mixing energy hypersurface, for any $1 \leq i \leq I_e$. Therefore, the Hamiltonian level (H, e) is topologically mixing. $\square$

CONCLUSIONS AND FUTURE WORK

This thesis is a contribution to the conservative and Hamiltonian dynamical systems theory.

The first results are on Hamiltonian dynamics and are published in the paper *"On the stability of the set of hyperbolic closed orbits of a Hamiltonian"*, co-authored with Mário Bessa and Jorge Rocha (see [13]). This work is related with a Mañé's conjecture, whereby any star system has its non-wandering set hyperbolic (see [50]). Several authors have been proving results on this conjecture: Mañé, for diffeomorphisms (see [53]), Gan and Wen, for vector fields (see [37]), and Bessa and Rocha, for divergence-free vector fields defined on a 3-dimensional manifold (see [20]). In this paper, we study this conjecture for Hamiltonian vector fields defined on a 4-dimensional symplectic manifold. The biggest challenge was to correctly formulate and adapt the definitions to this new context. We show that a *Hamiltonian star system* is *Anosov* and then that a C^2-*structurally stable Hamiltonian system* is *Anosov*. Moreover, we prove the *openness* and the *structural stability* of Anosov Hamiltonian systems defined on a $2d$-dimensional manifold, for $d \geq 2$.

The second problem is also related with the Mañé conjecture, mentioned above, but now for divergence-free vector fields defined on manifolds with dimension greater than 3. This work, entitled *"Stability properties of divergence-free vector fields"*, is a generalization, for any dimension, of the results in [20] and is available as a preprint (see [34]). We conclude that a divergence-free *star* vector field and that a *structurally stable* divergence-free vector field are, in fact, *Anosov* divergence-free vector fields. Moreover, we describe a general scenario for conservative dynamics in high dimensions. Now, we know that any divergence-free vector field can always be C^1-approximated by an

Anosov divergence-free vector field, or else by a divergence-free vector field exhibiting a *heterodimensional cycle*.

Some results concerning about *shadowing*, *Lipschitz shadowing* and *expansiveness* properties have been emerging, following the results of Mañé for diffeomorphism in [51]. We emphasize the works of Sakai, for diffeomorphism (see [73]), and of Moriyasu, Sakai and Sun and of Lee and Sakai, for vector fields (see [57, 47]). We contribute to these schemes by showing that a divergence-free vector field in the C^1-interior of the set of divergence-free vector fields satisfying the *shadowing property* is Anosov. The same conclusion is derived if the divergence-free vector field is taken in the C^1-interior of the set of divergence-free vector fields satisfying the *Lipschitz shadowing property* or in the C^1-interior of the set of expansive divergence-free vector fields. These results are contained in the paper, *"Shadowing, expansiveness and stability of divergence-free vector fields"*, available as a preprint (see [33]).

The last result is a generalization, to the Hamiltonian context, of a theorem due to Bonatti and Crovisier, which states that, C^1-generically, a conservative diffeomorphism is transitive (see [24]). This result was also extended for C^1-symplectic diffeomorphisms defined on a symplectic manifold (see [8]) and for divergence-free vector fields (see [11]). Our contribution is on to show that, for a C^2-generic Hamiltonian H, there exists an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that, for any $e \in \mathcal{S}(H)$, any connected component of $H^{-1}(\{e\})$ is *topologically mixing*. An important step to obtain this result is the formulation and proof of the *connecting lemma for pseudo-orbits of Hamiltonians*, which we also state.

Beyond the results proved in this thesis, there are other problems to explore in the future.

We expect that the results stated in the paper *"On the stability of the set of hyperbolic closed orbits of a Hamiltonian"* can, perhaps, be improved, by generalizing the results for Hamiltonians defined on $2d$-dimensional symplectic manifold, for $d > 2$.

We also would like to show if a Hamiltonian in the C^2-interior of the sets of Hamiltonians satisfying the *shadowing* property, or the *Lipschitz shadowing* property, or the *expansiveness* property is an *Anosov* Hamiltonian system. Again, an important steep to prove these results is the statement of proper definitions for the Hamiltonian context.

There are also a lot of interesting dichotomic results that are not yet proven for Hamiltonian systems. We emphasize the *Newhouse dichotomy* (see [71]) and the *Mañé-Bochi dichotomy* (see [23]). We remark that the generalization of the Newhouse dichotomy for Hamiltonians requires the transitivity property to be generic, which is already proved in this thesis.

To finish, a very attractive project is to prove the *generic ergodicity* among *partial hyperbolic Hamiltonians*, by generalizing the results in [10].

These are just some projects that can contribute to the development of the Hamiltonian theory.

We hope you find these results interesting and we expect to keep going on contributing to the development of the conservative and Hamiltonian dynamical systems theory.

APPENDIX

In this additional chapter, we state a different proof of the Lemma 2.6, in Chapter 2.2.1. This proof uses a generalization, for the high-dimensional context, of the adopted techniques in the proof of Lemma 3.1 in [20]. However, by Lemma 2.5, we already know that a vector field in $\mathcal{G}^1_\mu(M)$ has not singularities.

Lemma 2.6 If $X \in \mathcal{G}^1_\mu(M)$ then P^t_X admits a dominated splitting over M.

Proof: Consider $X \in \mathcal{G}^1_\mu(M)$ and a C^1-neighborhood $\mathcal{U}$ of X in $\mathcal{G}^1_\mu(M)$, small enough such that the dichotomy in Theorem 2.1 holds. By Lemma 2.5, we have that M is regular. Thus, P^t_X is well defined on M and there exists $\mathcal{V} \subset \mathcal{U}$, a C^1-neighborhood of X in $\mathcal{G}^1_\mu(M)$, whose elements do not have singularities. By contradiction, assume that M does not admit a dominated splitting. In this case, we claim that

Claim 3.3 *For any $\ell \in \mathbb{N}$, there exists a measurable, X^t-invariant set $\Gamma_\ell \subset M$ such that $\mu(\Gamma_\ell) > 0$ and Γ_ℓ does not have an ℓ-dominated splitting for P^t_X.*

In fact, if this claim is not true, there exists $\ell \in \mathbb{N}$ such that M has an ℓ-dominated splitting for P^t_X, which contradicts our assumption.

The existence of these sets Γ_ℓ without an ℓ-dominated splitting ($\ell \in \mathbb{N}$), allows us to use the techniques developed in the proof of [17, Theorem 1], where the authors show that, for any $\epsilon > 0$, there exists a large enough $\ell \in \mathbb{N}$ such that, for any arbitrarily small $\eta > 0$ and for μ-almost every point $x \in \Gamma_\ell$, we can find $t_0 > 0$ and $X_1 \in \mathcal{U}$, ϵ-C^1-close to X, satisfying

$$exp(-\eta t) < \|P^t_{X_1}(x)\| < exp(\eta t), \quad \forall\, t > t_0.$$

Now, let $R \subset \Gamma_\ell$ be the full μ-measure set of recurrent points with respect to X_1, given by the Poincaré Recurrence Theorem, and let $\mathcal{Z}_\eta \subset \Gamma_\ell$ be the set of points with Lyapunov exponent, associated to X_1, less than η. Therefore, fixing $\delta \in \left(0, \frac{\log 2}{(n-1)\ell}\right)$ and $\eta < \delta$, given $x \in \mathcal{Z}_\eta \cap R$, there exists $t_x \in \mathbb{R}$ such that

$$exp(-\delta t) < \|P^t_{X_1}(x)\| < exp(\delta t), \quad \forall\, t > t_x.$$

Now, once $x \in \mathcal{Z}_\eta \cap R$, by the volume-preserving Closing Lemma (Theorem 2.4), the X_1^t-orbit of x can be approximated by a closed orbit γ, with period π, of a C^1-close vector field $X_2 \in \mathcal{U}$. So, letting $r > 0$ be small enough in Theorem 2.4, $\tau > 0$ as in Theorem 2.1 and fixing $p \in \gamma$, we can choose $\pi > \tau$, arbitrarily large, such that

$$\exp(-\delta\pi) < \|P^\pi_{X_2}(p)\| < \exp(\delta\pi). \tag{3.3}$$

Recall that, since $X_2 \in \mathcal{U}$, the X_2-closed orbit γ with period $\pi > \tau$ is hyperbolic. Hence, by Theorem 2.1, there is $\ell_0 > 0$ such that $P^t_{X_2}$ admits an ℓ_0-dominated splitting $N_q = N_q^1 \oplus \cdots \oplus N_q^k$, for $2 \leq k \leq n-1$, such that

$$\|P^{\ell_0}_{X_2}(q)|_{N_q^i}\| \cdot \|P^{-\ell_0}_{X_2}(q)|_{N_q^j}\| \leq \frac{1}{2},$$

for every $0 \leq i < j \leq k$ and for every $q \in \mathcal{O}_{X_2}(p)$.

Now, as p is a hyperbolic saddle with period π for X_2, we can assume that $P^\pi_{X_2}(p)$ admits the following Lyapunov spectrum:

$$\lambda_1(p) \geq ... \geq \lambda_r(p) > 0 > \lambda_{r+1}(p) \geq ... \geq \lambda_k(p).$$

So, let $N_p^u := N_p^1 \oplus \cdots \oplus N_p^r$ and $N_p^s := N_p^{r+1} \oplus \cdots \oplus N_p^k$.

Let $[a]$ denote the integer part of a and observe that

$$\begin{aligned}
&\|P^\pi_{X_2}(p)|_{N_p^s}\| \cdot \|P^{-\pi}_{X_2}(p)|_{N_p^u}\| \\
&= \|P^{\pi-\ell_0[\pi/\ell_0]+\ell_0[\pi/\ell_0]}_{X_2}(p)|_{N_p^s}\| \cdot \|P^{-\pi-\ell_0[\pi/\ell_0]+\ell_0[\pi/\ell_0]}_{X_2}(p)|_{N_p^u}\| \\
&\leq \|P^{\pi-\ell_0[\pi/\ell_0]}_{X_2}(p)|_{N_p^s}\| \cdot \|P^{\ell_0[\pi/\ell_0]}_{X_2}(X_2^{\ell_0[\pi/\ell_0]}(p))|_{N^s_{X_2^{\ell_0[\pi/\ell_0]}(p)}}\| \cdot \\
&\quad \cdot \|P^{-\pi+\ell_0[\pi/\ell_0]}_{X_2}(p)|_{N_p^u}\| \cdot \|P^{-\ell_0[\pi/\ell_0]}_{X_2}(X_2^{-\ell_0[\pi/\ell_0]}(p))|_{N^u_{X_2^{-\ell_0[\pi/\ell_0]}(p)}}\|
\end{aligned}$$

$$
\begin{aligned}
&\le C(p, X_2) \prod_{i=1}^{[\pi/\ell_0]} \|P_{X_2}^{\ell_0}(X_2^{\ell_0}(p))|_{N^s_{X_2^{i\ell_0}(p)}}\| \cdot \|P_{X_2}^{-\ell_0}(X_2^{-\ell_0}(p))|_{N^u_{X_2^{-i\ell_0}(p)}}\| \\
&\le C(p, X_2) \left(\frac{1}{2}\right)^{[\pi/\ell_0]}, \qquad (3.4)
\end{aligned}
$$

where $C(p, X_2) = \sup_{0\le t\le \ell_0} \left(\|P_{X_2}^t(p)|_{N_p^s}\| \cdot \|P_{X_2}^{-t}(p)|_{N_p^u}\|\right)$. Since $C(p, X_2)$ depends continuously on X_2, in the C^1-topology, there exists a uniform bound for $C(p,\cdot)$, for every vector field which is C^1-close to X_2.

As mentioned in Remark 3, we have that $\sum_{i=1}^{k} \lambda_i(p) = 0$. Then, recalling that $\|P_{X_2}^{\pi}(p)\| = \|P_{X_2}^{\pi}(p)|_{N_p^u}\|$, we have that

$$
\begin{aligned}
\frac{1}{\pi}\log\|P_{X_2}^{\pi}(p)|_{N_p^1}\| = \lambda_{r+1}(p) &= -\sum_{\substack{i=1\\ i\neq r+1}}^{k} \lambda_i(p) \\
&\ge -(k-1)\lambda_1(p) = \frac{-(k-1)}{\pi}\log\|P_{X_2}^{\pi}(p)|_{N_p^u}\| \\
&= \frac{-(k-1)}{\pi}\log\|P_{X_2}^{\pi}(p)\|.
\end{aligned}
$$

Therefore, given that $\|P_{X_2}^{\pi}(p)|_{N_p^u}\|^{-1} \le \|P_{X_2}^{-\pi}(p)|_{N_p^u}\|$, from (3.4), we have that

$$
\begin{aligned}
&\|P_{X_2}^{\pi}(p)|_{N_p^s}\|\|P_{X_2}^{\pi}(p)|_{N_p^u}\|^{-1} \le C(p, X_2)\left(\frac{1}{2}\right)^{[\pi/\ell_0]} \\
\Leftrightarrow &\log\|P_{X_2}^{\pi}(p)|_{N_p^s}\| - \log\|P_{X_2}^{\pi}(p)|_{N_p^u}\| \le \log C(p, X_2) - [\pi/\ell_0]\log 2 \\
\Leftrightarrow &\frac{1}{\pi}\log\|P_{X_2}^{\pi}(p)\| \ge -\frac{\log C(p, X_2)}{\pi} + \frac{[\pi/\ell_0]\log 2}{\pi} + \frac{1}{\pi}\log\|P_{X_2}^{\pi}(p)|_{N_p^s}\| \\
\Leftrightarrow &\frac{1}{\pi}\log\|P_{X_2}^{\pi}(p)\| \ge -\frac{\log C(p, X_2)}{\pi} + \frac{[\pi/\ell_0]\log 2}{\pi} \\
&\qquad\qquad - \frac{(k-1)}{\pi}\log\|P_{X_2}^{\pi}(p)\|.
\end{aligned}
$$

Now, taking π arbitrarily large,

$$
\frac{1}{\pi}\log\|P_{X_2}^{\pi}(p)\| \ge \frac{\log 2}{k\ell_0} \ge \frac{\log 2}{(n-1)\ell_0} > \delta.
$$

But this contradicts (3.3). Thus P_X^t admits a dominated splitting over M. $\square$

BIBLIOGRAPHY

[1] F. Abdenur, A. Avila and J. Bochi. Robust transitivity and topological mixing for C^1-flows. *Proc. Amer. Math. Soc.* **132**, **3** (2003) 699–705.

[2] R. Abraham and J.E. Marsden. *Foundations of Mechanics*. The Benjamin/Cummings Publishing Company. Advanced Book Program, 2^{nd} edition (1980).

[3] W. Ambrose and S. Kakutani. Structure and continuity of measurable flows. *Duke Math. J.* **9** (1942) 25-42.

[4] A. Andronov and L. Pontrjagin. Systémes grossiers. *Dokl. Akad. Nauk. SSSR.* **14** (1937) 247–251.

[5] D.V. Anosov. Geodesic flows on closed Riemannian manifolds of negative curvature. *Proc. Steklov Math. Inst.* **90** (1967) 1–235.

[6] A. Arbieto and C. Matheus. A pasting lemma and some applications for conservative systems. *Erg. Th. & Dyn. Syst.* **27** (2007) 1399–1417.

[7] M.-C. Arnaud. Un lemme de fermeture d'orbites: le "orbit closing lemma". [An orbit closing lemma]. *C. R. Acad. Sci. Paris Sér. I Math.* **323**, **11** (1996) 1175–1178.

[8] M.-C. Arnaud, C. Bonatti and S. Crovisier. Dynamique sympletiques génériques. *Erg. Th. & Dyn. Syst.* **25**, **5** (2005) 1401–1436.

[9] A. Arroyo and R. Hertz. Homoclinic bifurcation and uniform hyperbolicity for three-dimensional flows. *Ann. I. H. Poincaré AN* **20**, **5** (2003) 805–841.

[10] A. Avila, J. Bochi and A. Wilkinson. Nonuniform center bunching and the genericity of ergodicity among partially hyperbolic symplectomorphisms. *Annales Scientifiques de l'École Normale Supérieure* **42**, **6** (2009) 931–979.

[11] M. Bessa. A generic incompressible flow is topological mixing. *Comptes Rendus Mathematique* **346** (2008) 1169–1174.

[12] M. Bessa. The Lyapunov exponents of generic zero divergence-free three-dimensional vector fields. *Erg. Th. & Dyn. Syst.* **27**, **6** (2007) 1445–1472.

[13] M. Bessa, C. Ferreira and J. Rocha. On the stability of the set of hyperbolic closed orbits of a Hamiltonian. *Math. Proc. Camb. Phil. Soc.* **149**, **2** (2010) 373–383.

[14] M. Bessa and J. Lopes Dias. Generic Dynamics of 4-Dimensional C^2 Hamiltonian Systems. *Commun. in Math. Phys.* **281** (2008) 597–619.

[15] M. Bessa and J. Lopes Dias. Hamiltonian elliptic dynamics on symplectic 4-manifolds. *Proc. Amer. Math. Soc.* **137** (2009) 585–592.

[16] M. Bessa and J. Rocha. Expansiveness, specification, hyperbolicity and heterodimensional cycles for conservative maps. Preprint (2012).

[17] M. Bessa and J. Rocha. Contributions to the geometric and ergodic theory of conservative flows. Preprint (2008), arXiv:0810.3855.

[18] M. Bessa and J. Rocha. Homoclinic tangencies versus uniform hyperbolicity for conservative 3–flows. *J. Differential Equations* **247** (2009) 2913–2923.

[19] M. Bessa and J. Rocha. On C^1-robust transitivity of volume-preserving flows. *J. Differential Equations* **245**, **11** (2008) 3127–3143.

[20] M. Bessa and J. Rocha. Three-dimensional conservative star flows are Anosov. *Disc. & Cont. Dynam. Sys. A* **26**, **3** (2010) 839–846.

[21] G. Birkhoff. Collected Mathematical Papers **1-3** (New York, 1950).

[22] G. Birkhoff. *Dynamical Systems*. Amer. Math. Soc. (Providence, RI, 1927).

[23] J. Bochi. C^1-generic symplectic diffeomorphisms: partial hyperbolicity and zero center Lyapunov exponents. *Journal of the Inst. of Math. Jussieu* **9**, **1** (2010) 49–93.

[24] C. Bonatti and S. Crovisier. Récurrence et généricité. *Invent. Math.* **158**, **1** (2004) 33–104.

[25] C. Bonatti, L. Díaz and M. Viana. *Dynamics beyond uniform hyperbolicity* (Springer-Verlag, New York, 2005, 288–289).

[26] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes in Math* (Springer-Verlag, Berlin, 1975).

[27] R. Bowen. *On Axiom A Diffeomorphisms* (Regional Conference Series in Mathematics, 35). Conference Board of the Mathematical Sciences (Providence, 1978).

[28] R. Bowen and P. Walters. Expansive one-parameter flows. *J. Differential Equations* **12** (1972) 180–193.

[29] M. Brin and G. Stuck. *Introduction to Dynamical Systems* (Cambridge University Press, 2002).

[30] J. Moser and E. Zehnder. *Notes on dynamical systems*. Courant Lecture Notes in Mathematics, 12. New York University, Courant Institute of Mathematical Sciences. Amer. Math. Soc. (Providence, RI, 2005).

[31] H. Ding. Disturbance of the homoclinic trajectory and applications. *Acta Sci. Nat. Univ. Pekin* **1** (1986) 53–63.

[32] C. Doering. Persistently transitive vector fields on three-dimensional manifolds. *Proc. Dynam. Sys. & Bifurc. Th.* Pitman Res. Notes in Math. **160** (1987) 59–89.

[33] C. Ferreira. Shadowing, expansiveness and stability of divergence-free vector fields. Preprint (2010), arXiv:1011.3546.

[34] C. Ferreira. Stability properties of divergence-free vector fields. (to appear) Dynamical Systems: An International Journal.

[35] J. Franks. Necessary conditions for stability of diffeomorphisms. *Trans. Amer. Math. Soc.* **158** (1971) 301–308.

[36] S. Gan. The star systems $\mathfrak{X}^*$ and a proof of the C^1 Ω-stability conjecture for flows. *J. Differential Equations* **163**, **1** (2000) 1–17.

[37] S. Gan and L. Wen. Nonsingular star flows satisfy Axiom A and the no-cycle condition. *Invent. Math.* **164** (2006) 279–315.

[38] W.H. Gottschalk and G.A. Hedlund. *Topological Dynamics.* Amer. Math. Soc. Colloq. Publ. **36** (Providence, RI, 1995).

[39] J. Guchenheimer. *A strange, in strange attractor, in The Hopf Bifurcation Theorems and Its Applications.* Appl. Math. Ser., Vol. 19, Springer-Verlag, Berlin and New York, 1976, 368–381.

[40] C. Gutierrez. A counter-example to a C^2 closing lemma. *Erg. Th. & Dyn. Syst.* **7** (1987) 509–530.

[41] S. Hayashi. Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows. *Ann. of Math.* (**2**) **145**, **1** (1997) 81–137.

[42] S. Hayashi. Correction to: 'Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows'. *Ann. of Math.* (**2**) **150**, **1** (1999) 353-356.

[43] S. Hayashi. Diffeomorphisms in $\mathcal{F}^1(M)$ satisfy Axiom A. *Ergod. Th. & Dynam. Sys.* **12**, **2** (1992) 233–253.

[44] M.W. Hirsch, C.C. Pugh and M. Shub. *Invariant manifolds. Lecture Notes in Mathematics* **583** (Springer-Verlag, 1977).

[45] W. Hurewicz and H. Wallman. *Dimension Theory. Princeton Mathematical Series* **4** (Princeton University Press, Princeton, NJ, 1941).

[46] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems.* (Cambridge University Press, 1995).

[47] K. Lee and K. Sakai. Structural stability of vector fields with shadowing. *J. Differential Equations* **232** (2007) 303–313.

[48] M. Li, S. Gan and L. Wen. Robustly transitive singular sets via approach of an extended linear Poincaré flow. *Disc. & Cont. Dynam. Sys.* **13**, **2** (2005) 239–269.

[49] C. Li and L. Wen. χ^* plus Axiom A does not imply no-cycle. *J. Differential Equations* **119**, **2** (1995) 395-400.

[50] R. Mañé. A proof of the C^1 stability conjecture. *Inst. Hautes Études Sci. Publ. Math.* **66** (1988) 161–210.

[51] R. Mañé. An ergodic closing lemma. *Ann. of Math.* **116**, **3** (1982) 503–540.

[52] R. Mañé. Contributions to the stability conjecture. *Topology* **17** (1978) 383–396.

[53] R. Mañé. Quasi-Anosov diffeomorphisms and hyperbolic manifolds. *Trans. Amer. Math. Soc.* **229** (1977) 351–370.

[54] Y. Matsumoto. *An Introduction to Morse Theory*. (American Mathematical Society, 2002).

[55] K. Menger. Ueber die Dimensionalität von Punktmengen I. *Monatshefte Math. und Phys.* **33** (1923) 148-160.

[56] C.A. Morales, M.J. Pacifico and R. Pujals. Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers. *Ann. of Math.* **160** (2004) 375–432.

[57] K. Moriyasu, K. Sakai and W. Sun. C^1-stably expansive flows. *J. Differential Equations* **213** (2005) 352–367.

[58] J. Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.* **120** (1965) 286–294.

[59] V.I. Oseledets. A multiplicative ergodic theorem: Lyapunov characteristic number for dynamical systems. *Trans. Moscow Math. Soc.* **19** (1968) 197–231.

[60] J. Palis. Open questions leading to a global perspective in dynamics. *Nonlinearity* **21** (2008) 37-43.

[61] J. Palis and S. Smale. Structural stability theorems. *Proc. Sympos. Pure Math.*, XIV, A.M.S., (1970) 223-231.

[62] J. Palis and F. Takens. *Hyperbolicity and Sensitive Chaotic Dynamics of Homoclinic Bifurcations* (Cambridge Univ. Press, Cambridge, 1993).

[63] S. Pilyugin. *Shadowing in Dynamical Systems. Lecture Notes Math* (Springer, Berlin, 1999).

[64] S. Pilyugin and S. Tikhomirov. Lipschitz shadowing implies structural stability. *Nonlinearity* **23** (2010) 2509–2515.

[65] C. Pugh. Against the C^2 closing lemma. *J. Differential Equations* **17** (1975) 435–443.

[66] C. Pugh and C. Robinson. The C^1 closing lemma, including Hamiltonians. *Ergod. Th & Dynam. Sys.* **3** (1983) 261–313.

[67] E. Pujals and M. Sambarino. Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. *Ann. of Math.* **2**, **151**, **3** (2000) 961–1023.

[68] E. Pujals and M. Sambarino. On the dynamic of dominated splitting. *Ann. of Math.* **2**, **169**, **3** (2009) 675–739.

[69] C. Robinson. Generic properties of conservative systems. *Amer. J. Math.* **92** (1970) 562–603.

[70] C. Robinson. *Lectures on Hamiltonian Systems. Monografias de Matemática*. (IMPA, Rio de Janeiro, 1971).

[71] R. Saghin and Z. Xia. Partial Hyperbolicity or dense elliptical periodic points for C^1-generic symplectic diffeomorphisms. *Trans. Amer. Math. Soc.* **358**, **11** (2006) 5119–5138.

[72] M. Shub. *Global Stability of Dynamical Systems* (Springer-Verlag, New York, 1987).

[73] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967) 747–817.

[74] S. Tikhomirov. Interiors of sets of vector fields with shadowing properties that correspond to some classes of reparametrizations. *Vestnik St. Petersburg Univ. Math.* **41**, **4** (2008) 360-366.

[75] H. Toyoshiba. Vector fields in the interior of Kupka-Smale systems satisfy axiom A. *J. Differential Equations* **177** (2001) 27–48.

[76] P.S. Urysohn. Les multiplicités cantoriennes. *C.R. Acad. Sci.* **175** (1922) 440-442.

[77] T. Vivier. Projective hyperbolicity and fixed points. *Erg. Th. & Dyn. Syst.* **26** (2006) 923–936.

[78] T. Vivier. Robustly transitive 3-dimensional regular energy surfaces are Anosov. *Institut de Mathématiques de Bourgogne*, Dijon, Preprint 412 (2005). http://math.u-bourgogne.fr/topo/prepub/pre05.html.

[79] L. Wen and Z. Xia. C^1 connecting lemmas. *Trans. Amer. Math. Soc.* **352** (2000) 5213–5230.

[80] Z. Xia and H. Zhang. A C^r closing lemma for a class of symplectic diffeomorphisms. *Nonlinearity* **19** (2006) 511–516.

[81] C. Zuppa. Regularisation C^∞ des champs vectoriels qui préservent l'elément de volume. *Bol. Soc. Bras. Mat.* **10**, **2** (1979) 51–56.